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The partition function of the supersymmetric two-dimensional black hole and little string theory*

Dan Israël,[a] Costas Kounnas,[a] Ari Pakman[ab] and Jan Troost[a]

[a] *Laboratoire de Physique Théorique de l'École Normale Supérieure*[†]
24, Rue Lhomond 75231 Paris Cedex 05, France, and
[b] *Racah Institute of Physics, The Hebrew University*
Jerusalem 91904, Israel
E-mail: israel@lpt.ens.fr, kounnas@lpt.ens.fr, pakman@phys.huji.ac.il, troost@lpt.ens.fr

Abstract: We compute the partition function of the supersymmetric two-dimensional euclidean black hole geometry described by the $SL(2,\mathbb{R})/U(1)$ superconformal field theory. We decompose the result in terms of characters of the $N=2$ superconformal symmetry. We point out puzzling sectors of states besides finding expected discrete and continuous contributions to the partition function. By adding an $N=2$ minimal model factor of the correct central charge and projecting on integral $N=2$ charges we compute the partition function of the background dual to little string theory in a double scaling limit. We show the precise correspondence between this theory and the background for NS5-branes on a circle, due to an exact description of the background as a null gauging of $SL(2,\mathbb{R}) \times SU(2)$. Finally, we discuss the interplay between GSO projection and target space geometry.

Keywords: String Duality, Superstring Vacua, Conformal Field Models in String Theory, Black Holes in String Theory.

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[†]Unité mixte du CNRS et de l'Ecole Normale Supérieure, UMR 8549.

Contents

1. Introduction

Conformal field theories with an $N = 2$ superconformal symmetry have proven to be one of the most important building blocks in superstring theory backgrounds. Firstly, a theorem of Banks and Dixon [1] states than any superstring vacua with (at least) $N = 1$ spacetime supersymmetry in four dimensions is built by combining internal conformal field theories with worldsheet $N = 2$ superconformal symmetry.[1] A large class of examples is provided by Gepner models [3], that use the minimal $N = 2$ theories — constructed with the $\mathrm{SU}(2)/\mathrm{U}(1)$ coset — to obtain non-trivial supersymmetric string compactifications. The $N = 2$ algebra was also used to study Calabi-Yau compactifications in general, leading in particular to the further development of mirror symmetry [4, 5].

[1] See [2] for a generalization to $Ads_3 \times \mathcal{M}$ backgrounds.

The non-minimal $N = 2$ theories have been shown in [6] to be intimately related to the (bosonic) coset theory $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$. In this paper we will elaborate on this connection by identifying these non-minimal $N = 2$ representations with the coset representations of the supersymmetric $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ coset [8], i.e. the gauged supersymmetric WZW model. The axial coset has a two-dimensional target space with a cigar geometry, and in two dimensions it has an interpretation as a euclidean black hole [7]. Recent interest for this superconformal field theory derives from the duality with the $N = 2$ super-Liouville theory, conjectured in [9] and studied in [10, 11]. The present work casts some new light on this duality, because we show explicitely that the partition function of the supersymmetric $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ can be decomposed in terms of (extended) $N = 2$ characters, which are the buildings blocks of the $N = 2$ super-Liouville theory [12]. This is natural in the light of the fact that the characters for the two models are identical. Another motivation for this work comes from the double scaling limit of the little string theory [9], corresponding to NS5-branes spread on a (topologically trivial) circle in the transverse space. In [9] the worldsheet CFT corresponding to the bulk geometry was argued to be the orbifold

$$\frac{\mathrm{SL}(2,\mathbb{R})_k/\mathrm{U}(1) \times \mathrm{SU}(2)_k/\mathrm{U}(1)}{\mathbb{Z}_k}.$$

In this work we compute the partition function for this theory, and show explicitly that the $\mathbb{Z}_k$ is completely fixed by the requirement of spacetime supersymmetry, without any a priori assumption on the bulk geometry. Then we argue that this projection changes drastically the low-energy interpretation of the background, and that this orbifold is actually equivalent to a null gauged WZW model:

$$\frac{\mathrm{SL}(2,\mathbb{R})_k \times \mathrm{SU}(2)_k}{\mathrm{U}(1)_L \times \mathrm{U}(1)_R},$$

whose target space corresponds precisely to a configuration of five-branes on a circle in supergravity, previously written down in [13].

Note added. When this work was near completion, the interesting work [56] appeared on the archive with a large overlap with the present paper. Among the complementary issues we consider are: the supersymmetric marginal deformation method, a precise analysis of the partition function and its decomposition in characters (including multiplicities of non-primary states) and the relationship with the background for five-branes on a circle. On the other hand [56] has other additional and interesting material especially regarding the elliptic genus for these non-compact models.

2. The supersymmetric coset by deformation

Our aim is to find the modular invariant torus partition function for the supersymmetric $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ coset conformal field theory. The strategy we will use is to deform the supersymmetric $\mathrm{SL}(2,\mathbb{R})_k$ WZW model by a marginal deformation that preserves supersymmetry. It is a very economic way of obtaining the supersymmetric partition function for the supersymmetric coset. As a byproduct, it is interesting on its own right since supersymmetric deformations of $\mathrm{SL}(2,\mathbb{R})$ have only been studied in the asymmetric case (see [16]). In the bosonic case, the partition function of the coset has been computed in [14]. The discrete part has been also studied in [15].

2.1 Partition function for supersymmetric $\mathrm{SL}(2,\mathbb{R})$

The partition function for the $N=1$ supersymmetric $\mathrm{SL}(2,\mathbb{R})$ at level k is made of the direct product of a modular invariant factor for the purely bosonic $\mathrm{SL}(2,\mathbb{R})$ at level $k+2$ and a modular invariant[2] factor for the free fermions, realizing a level -2 $\mathrm{SL}(2,\mathbb{R})$ algebra. The purely bosonic partition function has been obtained in [17] (following methods in [32, 14]). We refer the reader to these works for more detail. Let us start with the type $0B$ partition function (i.e. the partition function diagonal in fermion boundary conditions), for the universal cover of the $\mathrm{SL}(2,\mathbb{R})$ algebra — i.e. AdS_3:[3]

$$\begin{aligned} Z = {} & \frac{\sqrt{k\tau_2}}{\eta\bar{\eta}} \int d^2 s \frac{e^{2\pi\tau_2 s_1^2}}{|\vartheta_1(s_1\tau - s_2|\tau)|^2} \times \\ & \times \int_0^1 dt_1\, dt_2 \sum_{n,w\in\mathbb{Z}} e^{2i\pi n s_2} q^{\frac{1}{4(k+2)}(n-(k+2)(w+s_1-t_1))^2} \bar{q}^{\frac{1}{4(k+2)}(n+(k+2)(w+s_1-t_1))^2} \times \\ & \times \sum_{\tilde{n},\tilde{w}} e^{2i\pi(n-\tilde{n})t_2} q^{-\frac{1}{4(k+2)}(\tilde{n}+(k+2)(\tilde{w}+t_1))^2} \bar{q}^{-\frac{1}{4(k+2)}(\tilde{n}-(k+2)(\tilde{w}+t_1))^2} \times \\ & \times \frac{1}{2}\sum_{a,b=0}^{1} \vartheta\begin{bmatrix} a \\ b \end{bmatrix}(\tau,0)\, \bar{\vartheta}\begin{bmatrix} a \\ b \end{bmatrix}(\tau,0)\,. \end{aligned} \tag{2.1}$$

The first two lines are the partition function for the bosonic coset $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ while the third line corresponds to the extra time-like $\mathrm{U}(1)$ that extends the coset theory to $\mathrm{SL}(2,\mathbb{R})$. They are coupled through the $\mathbb{Z}$ orbifold given by the integral over t_1 and t_2.[4] The fermionic characters are defined as usual:

$$\vartheta\begin{bmatrix} a \\ b \end{bmatrix}(\tau,\nu) = \sum_{f\in\mathbb{Z}} q^{\frac{1}{2}\left(f+\frac{a}{2}\right)^2} e^{2i\pi\left(\nu+\frac{b}{2}\right)\left(f+\frac{a}{2}\right)}\,.$$

To proceed, we first integrate over t_2, and obtain the constraint $n=\tilde{n}$. Then we introduce the spectral flow quantum number: $w_+ = w + w'$. Finally we factorize the supersymmetric $\mathcal{J}^3\bar{\mathcal{J}}^3$ lattice associated to the currents at level k, that will eventually be removed by the gauging:

$$\begin{aligned} Z = {} & \frac{\sqrt{k\tau_2}}{\eta\bar{\eta}} \int_0^1 ds_1\, ds_2 \frac{e^{2\pi\tau_2 s_1^2}}{|\vartheta_1(s_1\tau - s_2|\tau)^2|^2} \int_0^1 dt_1 \sum_{n,w\in\mathbb{Z}} e^{2i\pi n s_2} \times \\ & \times q^{\frac{1}{4(k+2)}(n-(k+2)(w+s_1-t_1))^2} \bar{q}^{\frac{1}{4(k+2)}(n+(k+2)(w+s_1-t_1))^2} \times \end{aligned}$$

[2]Or modular covariant factor for superstrings.

[3]We have removed the contribution of the neutral fermions associated to left and right $\mathrm{U}(1)$ currents J^3 and $\bar{J}^3$ since they play no role in the following derivation.

[4]At least for integer level k, the partition function for the single cover is obtained by using instead a $\mathbb{Z}_{k+2}$ orbifold. But, as it is explained in [17], any cover of $\mathrm{SL}(2,\mathbb{R})$ gives the same axial coset.

$$
\begin{aligned}
&\times \sum_{f,\bar{f},w_+\in\mathbb{Z}} \frac{1}{2}\sum_{a,b\in\mathbb{Z}_2} e^{i\pi(f-\bar{f})b} \times \\
&\times q^{\frac{k}{2(k+2)}\left(f+\frac{a}{2}+\frac{n-(k+2)(w-t_1)+(k+2)w_+ +2f+a}{k}\right)^2} q^{-\frac{1}{k}\left(\frac{n}{2}-\frac{k+2}{2}(w-t_1)+\frac{k+2}{2}w_+ +f+\frac{a}{2}\right)^2} \times \\
&\times \bar{q}^{\frac{k}{2(k+2)}\left(\bar{f}+\frac{a}{2}+\frac{-n-(k+2)(w-t_1)+(k+2)w_+ +2\bar{f}+a}{k}\right)^2} \bar{q}^{-\frac{1}{k}\left(-\frac{n}{2}-\frac{k+2}{2}(w-t_1)+\frac{k+2}{2}w_+ +\bar{f}+\frac{a}{2}\right)^2} . \qquad (2.2)
\end{aligned}
$$

The total current of the elliptic sub-algebra is given by the sum of the purely bosonic current and the fermionic current: $\mathcal{J}^3 = J^3 + :\psi^+\psi^-:$, where the fermions have charge ± 1 under the global U(1).

2.2 Deformation towards the supersymmetric coset

A marginal deformation compatible with the (local) $N = 1$ superconformal symmetry of the supersymmetric $\mathrm{SL}(2,\mathbb{R})$ theory is given by the current-current operator constructed with the *total* currents of the Cartan sub-algebra.[5] The action is perturbed by the operator:

$$
\delta S \sim \int d^2 z \; \mathcal{J}^3(z)\bar{\mathcal{J}}^3(\bar{z}) .
$$

The action of this deformation at the level of the partition function is obtained by changing the radius of the corresponding lattice of $(\mathcal{J}^3, \bar{\mathcal{J}}^3)$, which we call (P_L, P_R) in the following. We replace:

$$
\begin{aligned}
\frac{1}{2}P_L^2 &\to -\frac{1}{4k}\left[\left(n+f-\bar{f}\right)\mathcal{R} + \frac{\left(-(k+2)(w-t_1)+(k+2)w_+ + f+\bar{f}+a\right)}{\mathcal{R}}\right]^2 \\
\frac{1}{2}P_R^2 &\to -\frac{1}{4k}\left[-\left(n+f-\bar{f}\right)\mathcal{R} + \frac{\left(-(k+2)(w-t_1)+(k+2)w_+ + f+\bar{f}+a\right)}{\mathcal{R}}\right]^2 .
\end{aligned}
$$

The modular invariance is ensured along the line of deformation since it is a $O(1,1,\mathbb{R})$ transformation of a toroidal lattice. This should be understood in a formal sense since it is a divergent sum over timelike momenta. To go from the parent theory $\mathrm{SL}(2,\mathbb{R})$ to the coset theory $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{U}(1)$ we have to remove all the contributions of the U(1) which is gauged. It is known, at least for the bosonic case [18, 17], that it can be done in a modular invariant way by taking an *infinite* marginal deformation, either $\mathcal{R} \to 0$ (axial coset) or $\mathcal{R} \to \infty$ (vector coset). In those limits the contribution of the extra U(1) decouples and can be safely removed. Here, since we would like to find the partition function for the cigar - the axial coset, we have to take the limit where the radius goes to zero. We have to *(i)* do an analytic continuation to get a well-defined sum over the momenta: $\mathcal{R} \to i\mathcal{R}$ and *(ii)* take the limit $\mathcal{R} \to 0$. The validity of this procedure will be checked later by showing explicitely the modular invariance of the resulting partition function. In the zero radius limit (i.e. axial coset), we get the constraint:

$$
-(k+2)(w-t_1) + (k+2)w_+ + f + \bar{f} + a = 0 . \qquad (2.3)
$$

[5]We have chosen as a Cartan the elliptic sub-algebra, in order to obtain at the end the euclidean black hole. For the minkowskian case one has to consider instead the hyperbolic (non-compact) one.

We define also $N = n + f - \bar{f}$. Then we get the following partition function, after dropping the decoupled contribution of the extra U(1):

$$
\begin{aligned}
Z = &\sqrt{k\tau_2}\,\frac{1}{2}\sum_{a,b\in\mathbb{Z}_2}\int_0^1 ds_1\,ds_2\,\frac{e^{2\pi\tau_2 s_1^2}}{|\vartheta_1(s_1\tau - s_2|\tau)^2|^2}\times\\
&\times\sum_{f,\bar{f},N,w_+\in\mathbb{Z}} e^{i\pi(f-\bar{f})b}e^{2i\pi(N-f+\bar{f})s_2}\times\\
&\times q^{\frac{1}{4(k+2)}(N-2f-a-(k+2)(w_++s_1))^2}\bar{q}^{\frac{1}{4(k+2)}\left(N+2\bar{f}+a+(k+2)(w_++s_1)\right)^2}\times\\
&\times q^{\frac{k}{2(k+2)}\left(f+\frac{a}{2}+\frac{N}{k}\right)^2}\bar{q}^{\frac{k}{2(k+2)}\left(\bar{f}+\frac{a}{2}-\frac{N}{k}\right)^2}.
\end{aligned}
\tag{2.4}
$$

This is the partition function for the supersymmetric coset $\mathrm{SL}(2,\mathbb{R})_k/\mathrm{U}(1)$ we wanted to compute. The central charge of this superconformal field theory is :

$$
c = \frac{3(k+2)}{k}.
$$

We recall that the N=2 R-current of the super-coset, obtained by the standard Kazama-Suzuki procedure [19] is:

$$
J_R =: \psi^+\psi^- : + \frac{2\mathcal{J}^3}{k}.
$$

The last line of the partition function, eq. (2.4) corresponds to the lattice of this $N = 2$ R-current by construction. So the left and right N=2 charges of the super-coset are:

$$
\mathcal{Q}_L = f + \frac{a}{2} + \frac{N}{k}, \qquad \mathcal{Q}_R = \bar{f} + \frac{a}{2} - \frac{N}{k}. \tag{2.5}
$$

2.3 Modular invariance

Our result is by construction modular invariant since it has been obtained as the limiting case of a line of modular invariant theories. Nevertheless we will check the modular invariance directly, by putting the partition function in a form that will be at any rate useful in the following.

Partition function in lagrangian form. First we factorize from the R-current lattice the contribution that came from the free fermions in the parent theory:

$$
\begin{aligned}
Z = &\sqrt{k\tau_2}\frac{1}{2}\sum_{a,b\in\mathbb{Z}_2}\int_0^1 ds_1\,ds_2\sum_{f,\bar{f},N,w_+\in\mathbb{Z}} e^{i\pi(f-\bar{f})b}\times\\
&\times e^{2i\pi(N-f+\bar{f})s_2}e^{-2i\pi\tau_1(N-f+\bar{f})s_1}e^{-2\pi\tau_2(f+\bar{f}+a+(k+2)w_+)s_1}e^{-\pi\tau_2 k s_1^2}\times\\
&\times\frac{1}{|\vartheta_1(s_1\tau - s_2|\tau)^2|^2}\, q^{\frac{1}{2}\left(f+\frac{a}{2}+w_+\right)^2+\frac{(N-kw_+)^2}{4k}}\bar{q}^{\frac{1}{2}\left(\bar{f}+\frac{a}{2}+w_+\right)^2+\frac{(N+kw_+)^2}{4k}}.
\end{aligned}
\tag{2.6}
$$

Then we introduce the flowed fermionic levels, since the spectral flow also has to act on the $\mathrm{SL}(2,\mathbb{R})$ algebra at level -2 made with the fermions, as explained in [20]:

$$
f + w_+ = \tilde{f}, \qquad \bar{f} + w_+ = \tilde{\bar{f}}.
$$

Recast in this form, namely eq. (2.6), the structure of the partition function is transparent. In fact, the weights that will appear in the spectrum will have the following form:

$$L_0 = L_0^{\mathrm{SL}(2,\mathbb{R}),bosonic} + L_0^{\mathrm{SL}(2,\mathbb{R}),fermionic} - L_0^{\mathrm{U}(1)_{-k}} \,,$$

which is exactly what is expected from the standard GKO construction [21].

To prove modular invariance, we observe that the partition function, (2.6) can be rewritten in the lagrangian form, by a Poisson resummation on the momenta N of the compact boson as follows:

$$\begin{aligned} Z = \frac{k}{2} \sum_{a,b\in\mathbb{Z}_2} \int_0^1 ds_1\, ds_2\, \frac{e^{2\pi\tau_2 s_1^2}}{|\vartheta_1(s_1\tau - s_2|\tau)^2|^2} \sum_{M,w_+\in\mathbb{Z}} e^{-\frac{k\pi}{\tau_2}|(w_+ + s_1)\tau - (M+s_2)|^2} \times \\ \times \vartheta\begin{bmatrix} a+2s_1 \\ b+2s_2 \end{bmatrix} \bar{\vartheta}\begin{bmatrix} a+2s_1 \\ b+2s_2 \end{bmatrix} . \end{aligned} \tag{2.7}$$

The modular invariance of this expression is straightforwardly checked since it has the structure of a freely acting (continuous) orbifold (see [17] for more details). The partition function (2.7) is clearly understood as the constrained product of $\mathrm{SL}(2,\mathbb{R})_{k+2} \times \mathrm{U}(1)_k \times \mathrm{U}(1)_2$.

The twisted super-coset partition function. The super-coset $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{U}(1)$ has a discrete chiral symmetry $\mathbb{Z}_2 \times \mathbb{Z}$. While the first factor corresponds to the usual $\mathbb{Z}_2$ symmetry of the fermions, the second factor is the symmetry of the non-compact supersymmetric parafermions. Any discrete subgroup of this $\mathbb{Z}$ symmetry can be gauged. We will need for the following the $\mathbb{Z}_k$ orbifold of this theory. Geometrically, this orbifold acts on the translational symmetry along the transverse compact coordinate of the cigar — the euclidean compact time direction of the 2D black hole. It is obtained by acting in the lattice of $\mathrm{U}(1)_k$ as follows:

$$\begin{aligned} Z_{BH} &= \frac{1}{k} \sum_{\gamma,\delta\in\mathbb{Z}_k} Z_{BH}\begin{bmatrix} \gamma \\ \delta \end{bmatrix} \\ &= \sum_{\gamma,\delta\in\mathbb{Z}_k} \frac{1}{2} \sum_{a,b\in\mathbb{Z}_2} \int_0^1 ds_1\, ds_2\, \frac{e^{2\pi\tau_2 s_1^2}}{|\vartheta_1(s_1\tau - s_2|\tau)^2|^2} \times \\ &\quad \times \sum_{M,w_+\in\mathbb{Z}} e^{-\frac{k\pi}{\tau_2}\left|\left(w_+ + s_1 + \frac{\gamma}{k}\right)\tau - \left(M + s_2 + \frac{\delta}{k}\right)\right|^2} \vartheta\begin{bmatrix} a+2s_1 \\ b+2s_2 \end{bmatrix} \bar{\vartheta}\begin{bmatrix} a+2s_1 \\ b+2s_2 \end{bmatrix} . \end{aligned} \tag{2.8}$$

The modular invariance of this expression is manifest. In the hamiltonian representation (i.e. after Poisson resummation), we will have in eq. 2.4: *(i)* to do the replacement $w_+ \to w_+ + \gamma/k$ and *(ii)* to insert the phase $e^{2i\pi N\delta/k}$. So we obtain the following twisted sectors:

$$\begin{aligned} Z_{BH}\begin{bmatrix} a;a \\ b;b \end{bmatrix}\begin{bmatrix} \gamma \\ \delta \end{bmatrix} &= \sqrt{k\tau_2} \int_0^1 ds_1\; ds_2\; \frac{e^{2\pi\tau_2 s_1^2}}{|\vartheta_1(s_1\tau - s_2|\tau)^2|^2} \times \\ &\quad \times \sum_{\tilde{f},\tilde{\bar{f}},N,w_+\in\mathbb{Z}} e^{i\pi(\tilde{f}-\tilde{\bar{f}})b} e^{2i\pi(N-\tilde{f}+\tilde{\bar{f}})s_2} \times \end{aligned}$$

$$\times q^{\frac{1}{4(k+2)}\left(N-2\tilde{f}-a-kw_+-\gamma-(k+2)s_1\right)^2}\bar{q}^{\frac{1}{4(k+2)}\left(N+2\tilde{\tilde{f}}+a+kw_++\gamma+(k+2)s_1\right)^2}\times$$

$$\times q^{\frac{k}{2(k+2)}\left(\tilde{f}+\frac{a}{2}+\frac{N-\gamma}{k}-w_+\right)^2}\bar{q}^{\frac{k}{2(k+2)}\left(\tilde{\tilde{f}}+\frac{a}{2}-\frac{N+\gamma}{k}-w_+\right)^2} e^{2i\pi N\frac{\delta}{k}}\,. \tag{2.9}$$

In this expression $\left[{a;\bar{a} \atop b;\bar{b}}\right]$ denotes the spin structures on the two-torus for the left and right-movers. If the left and right spin structures are not the same, one has to insert an additional phase $e^{-i\pi(b-\bar{b})s_1}$ to ensure modular invariance of the orbifold.

3. The character decomposition

The analysis of the partition function we obtained for the super-coset, eq. (2.4) is carried out by identifying the characters of the super-coset $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ corresponding to the different representations, and then the irreducible characters of the (non-minimal) $N=2$ superconformal algebra. In the following analysis, we will improve on the analyses that are present in the literature in the fact that we will take along multiplicities of all the states in the partition function explicitly.

3.1 The discrete representations

The first step in this decomposition is to expand the ϑ_1 factor coming from the twisted determinant of euclidean $AdS_3 = \mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(2)$ in powers of the holonomies of the gauge field. Explicitely, we have, with $y=e^{2i\pi(s_1\tau-s_2)}$:

$$\frac{1}{\vartheta_1(s_1\tau-s_2|\tau)}=\frac{i\ y^{1/2}q^{-1/12}}{\eta(\tau)}\frac{1}{\prod_{n=1}^{\infty}(1-yq^{n-1})(1-y^{-1}q^n)}=\frac{i\ y^{1/2}}{\eta^3}\sum_{r\in\mathbb{Z}}y^r S_r(\tau) \tag{3.1}$$

where the function S_r is defined by (see [23]):

$$S_r(\tau)=\sum_{n=0}^{\infty}(-)^n q^{\frac{n(n+2r+1)}{2}}\,. \tag{3.2}$$

Note that the above expansion, eq. (3.1) is valid only for $|q|<|y|<1$ which is indeed the case when $1>s_1>0$. We plug this expansion in the partition function, eq. (2.4), and we also *(i)* integrate over s_2 to get the constraint: $N=r-\bar{r}+\tilde{f}-\tilde{\tilde{f}}$ and *(ii)* introduce a gaussian integral over s to linearize the integral over s_1, as was done in [14] for the bosonic coset. Then we obtain the following expression:

$$\begin{aligned}Z=\ &(q\bar{q})^{-\frac{c-3}{24}}\frac{\tau_2}{\eta^3\bar{\eta}^3}\sum_{a,b\in\mathbb{Z}_2}\int_{-\infty}^{\infty}ds\int_0^1 ds_1\sum_{\tilde{f},\tilde{\tilde{f}},N,w_+\in\mathbb{Z}}e^{i\pi(\tilde{f}-\tilde{\tilde{f}})b}\times\\ &\times\sum_{r,\bar{r}\in\mathbb{Z}}e^{-2\pi\tau_2(2is+1+r+\bar{r}+\tilde{f}+\tilde{\tilde{f}}+a+kw_+)s_1}\delta_{r-\bar{r}+\tilde{f}-\tilde{\tilde{f}},N}\,S_r\bar{S}_{\bar{r}}\times\\ &\times q^{\frac{s^2+1/4}{k}+\frac{1}{4(k+2)}\left(-N+2\tilde{f}+a+kw_+\right)^2+\frac{k}{2(k+2)}\left(\tilde{f}+\frac{a}{2}+\frac{N-kw_+}{k}\right)^2}\times\\ &\times\bar{q}^{\frac{s^2+1/4}{k}+\frac{1}{4(k+2)}\left(N+2\tilde{\tilde{f}}+a+kw_+\right)^2+\frac{k}{2(k+2)}\left(\tilde{\tilde{f}}+\frac{a}{2}-\frac{N+kw}{k}\right)^2}\,.\end{aligned} \tag{3.3}$$

For the discrete representations first, we can proceed to the character decomposition. Starting from (3.3), we first integrate over s_1 to obtain:

$$
\begin{aligned}
Z = \; & \frac{1}{\pi}(q\bar{q})^{-\frac{c-3}{24}} \frac{1}{\eta^3\bar{\eta}^3} \frac{1}{2} \sum_{a,b\in\mathbb{Z}_2} \int_{-\infty}^{\infty} ds \sum_{\tilde{f},\tilde{\tilde{f}},N,w_+\in\mathbb{Z}} e^{i\pi(\tilde{f}-\tilde{\tilde{f}})b} \times \\
& \times \sum_{r,\bar{r}\in\mathbb{Z}} \delta_{r-\bar{r}+\tilde{f}-\tilde{\tilde{f}},N} \, S_r \bar{S}_{\bar{r}} \times \\
& \times q^{\frac{s^2+1/4}{k}+\frac{1}{4(k+2)}\left(-N+2\tilde{f}+a+kw_+\right)^2+\frac{k}{2(k+2)}\left(\tilde{f}+\frac{a}{2}+\frac{N-kw_+}{k}\right)^2} \times \\
& \times \bar{q}^{\frac{s^2+1/4}{k}+\frac{1}{4(k+2)}\left(N+2\tilde{\tilde{f}}+a+kw_+\right)^2+\frac{k}{2(k+2)}\left(\tilde{\tilde{f}}+\frac{a}{2}-\frac{N+kw}{k}\right)^2} \times \\
& \times \frac{1-e^{-2\pi\tau_2(2is+1+r+\bar{r}+\tilde{f}+\tilde{\tilde{f}}+a+kw_+)}}{2is+1+r+\bar{r}+\tilde{f}+\tilde{\tilde{f}}+a+kw_+} \, .
\end{aligned}
\tag{3.4}
$$

Discrete representations of the coset. The *discrete representations* of $\mathrm{SL}(2,\mathbb{R})$ are either highest weight or lowest weight. When we parameterize the Casimir as: $c_2 = s^2+1/4$, they are labeled by $s = i(1/2-j)$, where $j \in \mathbb{R}_+$ for the unitary representations. The spectrum of the elliptic sub-algebra (i.e. J^3) is: $m = \pm(j+r)$, $r \in \mathbb{N}$. We have also to consider *flowed representations* (see [24, 25]), for which the quantum numbers m equal $m = \tilde{m} + kw_+/2$.

To find the contribution of the discrete representations to the partition function, we observe that the second term in the last line of eq. (3.4) can be completed to a square if we shift the variable s as: $s \to s - ik/2$. We pick up residues between the two contours for:

$$2\Im s = 1 + r + \bar{r} + \tilde{f} + \tilde{\tilde{f}} + a + kw_+ \, .$$

These poles will correspond to the discrete representations of the coset. We find that they are located in the interval:

$$\frac{1}{2} < j < \frac{k+1}{2} \, , \tag{3.5}$$

which is precisely the improved unitarity range (for level $k+2$) for the discrete representations found in [25].

Representations $j = 1/2$ and $j = (k+1)/2$. The expression obtained after the integration over s_1, eq. (3.4), has a pole for $s = 0$ whenever we have:

$$1 + r + \bar{r} + \tilde{f} + \tilde{\tilde{f}} + a + kw_+ = 0 \, .$$

This pole corresponds to the discrete representation with $j = 1/2$. Thus we choose to define the integral over s with the principal value prescription (written as usual $P\int$), taking into account half of the pole. After the contour shift in the second term in the last line of eq. (3.4), we will find also half of the pole corresponding to the representation $j = (k+1)/2$.

JHEP06(2004)033

Discrete characters of the bosonic coset. To summarize, we can write the contribution of the discrete spectrum as:

$$
\begin{aligned}
Z_D = \frac{1}{\eta\bar{\eta}} \frac{1}{2} \sum_{a,b\in\mathbb{Z}_2} \int_{1/2}^{(k+1)/2} dj & \sum_{r,\bar{r},\tilde{f},\tilde{\bar{f}},N,w_+\in\mathbb{Z}} e^{i\pi(\tilde{f}-\tilde{\bar{f}})b}\, \lambda_r^j(q)\, \lambda_{\bar{r}}^j(\bar{q}) \times \\
& \times q^{\frac{k}{2(k+2)}\left(\tilde{f}+\frac{a}{2}+\frac{N-kw_+}{k}\right)^2} \bar{q}^{\frac{k}{2(k+2)}\left(\tilde{\bar{f}}+\frac{a}{2}+\frac{-N-kw_+}{k}\right)^2} \times \\
& \times \delta(2j+r+\bar{r}+kw_+ + \tilde{f} + \tilde{\bar{f}} + a)\, \delta_{r-\bar{r}+\tilde{f}-\tilde{\bar{f}},N}
\end{aligned} \tag{3.6}
$$

where the discrete characters of the bosonic coset $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ are defined as (for several derivations of these characters, see the references [26, 27, 28, 23, 29]):

$$
\lambda_r^j(q) = q^{-\frac{(j-1/2)^2}{k}+\frac{(j+r)^2}{k+2}} \frac{S_r(\tau)}{\eta^2(\tau)}\,. \tag{3.7}
$$

Note that we have found not only the correct conformal weights for the primaries, but also the right multiplicities for all the descendents contained in the characters. The discrete part of the partition function is thus composed of the product of discrete characters of the bosonic coset $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ and $\mathrm{U}(1)$ characters of the $N=2$ R-current, obeying some constraints. The constraints on the $N=2$ charges are those expected for the path integral construction of the supersymmetric coset [30], namely (m_t is the eigenvalue of $\mathcal{J}^3$):

$$
m_t + \bar{m}_t = \tilde{m}_t + \tilde{\bar{m}}_t + kw_+ = 2j + r + \bar{r} + kw_+ + \tilde{f} + \tilde{\bar{f}} + a = 0\,, \tag{3.8}
$$

and

$$
m_t - \bar{m}_t = r - \bar{r} + \tilde{f} - \tilde{\bar{f}} = N\,, \tag{3.9}
$$

The solution of these constraints is: $m_t = N/2$ and $\bar{m}_t = -N/2$, or, for the unflowed quantum numbers:

$$
\tilde{m}_t = \frac{N}{2} - \frac{k}{2}w_+ \quad \text{and} \quad \tilde{\bar{m}}_t = -\frac{N}{2} - \frac{k}{2}w_+. \tag{3.10}
$$

To proceed we can solve explicitely for the constraints:

- first we trade the integral over j by a sum over half-integer values within the range (3.5)

- then we solve the constraints for r and $\bar{r}$ for a given j:

$$
\begin{aligned}
r &= \frac{N}{2} - \frac{k}{2}w_+ - \left(\tilde{f} + \frac{a}{2}\right) - j \\
\bar{r} &= -\frac{N}{2} - \frac{k}{2}w_+ - \left(\tilde{\bar{f}} + \frac{a}{2}\right) - j
\end{aligned} \tag{3.11}
$$

Since r and $\bar{r}$ have to be integers, we will define in the following the discrete coset characters to be zero if the solutions of the constraints, eq. (3.11) are not integer-valued. With this convention in mind, we can recast the contribution of the discrete representations to the partition function as follows:

$$
\begin{aligned}
Z_D = {} & \frac{1}{\eta\bar{\eta}}(q\bar{q})^{-c/24}\frac{1}{2}\sum_{a,b\in\mathbb{Z}_2}\sum_{2j=1}^{k+1}\Upsilon(2j,1)\sum_{N,w_+\in\mathbb{Z}}\times \\
& \times\sum_{\tilde{f}\in\mathbb{Z}} e^{i\pi b\tilde{f}}\,\lambda^{j}_{\frac{N}{2}-\frac{k}{2}w_+-\left(\tilde{f}+\frac{a}{2}\right)-j}\; q^{\frac{k}{2(k+2)}\left(\tilde{f}+\frac{a}{2}+\frac{N-kw_+}{k}\right)^2}\times \\
& \times\sum_{\tilde{\tilde{f}}\in\mathbb{Z}} e^{-i\pi b\tilde{\tilde{f}}}\,\bar{\lambda}^{j}_{-\frac{N}{2}-\frac{k}{2}w_+-\left(\tilde{\tilde{f}}+\frac{a}{2}\right)-j}\; \bar{q}^{\frac{k}{2(k+2)}\left(\tilde{\tilde{f}}+\frac{a}{2}+\frac{-N-kw_+}{k}\right)^2}.
\end{aligned}
\tag{3.12}
$$

In this expression we have defined:

$$
\begin{aligned}
\Upsilon(2j,1) &= \frac{1}{2} \qquad \text{if } 2j=1 \mod k \\
&= 1 \quad \text{otherwise}.
\end{aligned}
\tag{3.13}
$$

Link with extended $N=2$ discrete characters. In this section we would like to make the connection to the decomposition of $N=2$ extended characters in term of the characters of the bosonic coset [31]. In fact, from this analysis it is clear that the extended $N=2$ characters and the extended characters for the supercoset agree. First, we observe that the partition function of the discrete representations, eq. (3.12), is written in terms of the following (twisted) unextended $N=2$ discrete characters:

$$
\begin{aligned}
ch_d(j,r)\begin{bmatrix}a\\b\end{bmatrix}(\tau,\nu) &= \frac{1}{\eta^3(\tau)}q^{-\frac{(j-1/2)^2+(j+r+a/2)^2}{k}}z^{\frac{2j+2r+a}{k}}\frac{\vartheta\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right](\tau,\nu)}{1+(-)^b z\, q^{(1+a)/2+r}} \\
&= \sum_{n\in\mathbb{Z}} e^{i\pi b\left(n+\frac{a}{2}\right)}\lambda^{j}_{\frac{k}{2}Q_m-n-\frac{a}{2}-j}\frac{q^{\frac{k}{2(k+2)}\left(n+\frac{a}{2}+Q_m\right)^2}}{\eta(\tau)}z^{n+\frac{a}{2}+Q_m}
\end{aligned}
\tag{3.14}
$$

where

$$
Q_m = \frac{2m}{k} = \frac{2(j+r)}{k}
$$

is the $N=2$ charge of the primary, and $z=e^{2i\pi\nu}$. Then we rewrite the discrete partition function as:

$$
\begin{aligned}
Z_D = {} & \frac{1}{2}\sum_{a,b\in\mathbb{Z}_2}\sum_{2j=1}^{k+1}\Upsilon(2j,1)\sum_{N,w_+\in\mathbb{Z}} ch_d\left(j,\frac{N}{2}-\frac{k}{2}w_+-j\right)\begin{bmatrix}a\\b\end{bmatrix}(\tau,0)\times \\
& \times\bar{ch}_d\left(j,-\frac{N}{2}-\frac{k}{2}w_+-j\right)\begin{bmatrix}a\\b\end{bmatrix}(\bar{\tau},0)
\end{aligned}
\tag{3.15}
$$

The extended characters of the $N=2$ algebra were introduced in [12] because of their nice modular properties. Their decomposition in terms of bosonic coset characters was given in [31]:[6]

$$
\mathrm{Ch}\,(j,r)\begin{bmatrix}a\\b\end{bmatrix} = \sum_{n'\in\mathbb{Z}} ch_d(j,r+kn')\begin{bmatrix}a\\b\end{bmatrix}(\tau,\nu)
\tag{3.16}
$$

[6]From now we assume that the level k is integer.

$$= \sum_{n,n' \in \mathbb{Z}} e^{i\pi b\left(n+\frac{a}{2}\right)} \lambda^{j}_{\frac{k}{2}Q_m - n - \frac{a}{2} - j + kn'} \frac{q^{\frac{k}{2(k+2)}\left(n+\frac{a}{2}+Q_m+2n'\right)^2}}{\eta(\tau)} z^{n+\frac{a}{2}+Q_m+2n'} .$$

They differ from the standard $N = 2$ characters, eq. (3.14) by a sum over even $N = 2$ charges. Now, starting from eq. (3.15), we define: $N = e + kp$. The charge e is defined modulo k. We obtain then:

$$\begin{aligned}
Z_D &= \frac{1}{2} \sum_{a,b \in \mathbb{Z}_2} \sum_{2j=1}^{k+1} \Upsilon(2j,1) \sum_{e \in \mathbb{Z}_k} \sum_{p,w_+ \in \mathbb{Z}} ch_d\left(j, \frac{e}{2} + \frac{k}{2}(p - w_+) - j\right) \begin{bmatrix} a \\ b \end{bmatrix} \times \\
&\quad \times \bar{ch}_d\left(j, -\frac{e}{2} - \frac{k}{2}(p + w_+) - j\right) \begin{bmatrix} a \\ b \end{bmatrix} \\
&= \frac{1}{2} \sum_{a,b \in \mathbb{Z}_2} \sum_{2j=1}^{k+1} \Upsilon(2j,1) \times \\
&\quad \times \sum_{e \in \mathbb{Z}_k} \left\{ \sum_{n,\bar{n}\ \mathrm{even}} ch_d\left(j, \frac{e}{2} + \frac{kn}{2} - j\right) \begin{bmatrix} a \\ b \end{bmatrix} \bar{ch}_d\left(j, -\frac{e}{2} - \frac{k\bar{n}}{2} - j\right) \begin{bmatrix} a \\ b \end{bmatrix} + \right. \\
&\qquad \left. + \sum_{n,\bar{n}\ \mathrm{odd}} ch_d\left(j, \frac{e}{2} + \frac{kn}{2} - j\right) \begin{bmatrix} a \\ b \end{bmatrix} \bar{ch}_d\left(j, -\frac{e}{2} - \frac{k\bar{n}}{2} - j\right) \begin{bmatrix} a \\ b \end{bmatrix} \right\}. \qquad (3.17)
\end{aligned}$$

So we get finally:

$$\begin{aligned}
Z_D &= \frac{1}{2} \sum_{a,b \in \mathbb{Z}_2} \sum_{2j=1}^{k+1} \Upsilon(2j,1) \times \\
&\quad \times \sum_{e \in \mathbb{Z}_k} \left\{ Ch_d\left(j, \frac{e}{2} - j\right) \begin{bmatrix} a \\ b \end{bmatrix} \bar{Ch}_d\left(j, -\frac{e}{2} - j\right) \begin{bmatrix} a \\ b \end{bmatrix} + \right. \\
&\qquad \left. + Ch_d\left(j, \frac{e+k}{2} - j\right) \begin{bmatrix} a \\ b \end{bmatrix} \bar{Ch}_d\left(j, -\frac{e+k}{2} - j\right) \begin{bmatrix} a \\ b \end{bmatrix} \right\} \\
&= \frac{1}{2} \sum_{a,b \in \mathbb{Z}_2} \sum_{2j=1}^{k+1} \Upsilon(2j,1) \sum_{2g \in \mathbb{Z}_{2k}} Ch_d(j, g - j) \begin{bmatrix} a \\ b \end{bmatrix} \bar{Ch}_d(j, -g - j) \begin{bmatrix} a \\ b \end{bmatrix}, \qquad (3.18)
\end{aligned}$$

where the charge $2g$ is defined modulo $2k$. In short, we have proven that, for k integer, the discrete spectrum of the supersymmetric coset $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ can be decomposed in terms of extended discrete characters of the $N = 2$ algebra.

3.2 The continuous representations

Now we would like to identify the contributions coming from the continuous representations of $\mathrm{SL}(2,\mathbb{R})$. These representations are labeled by the Casimir $c_2 = s^2 + 1/4$ with $s \in \mathbb{R}_+$, and their parity $2\alpha \in \mathbb{Z}_2$. These representations are neither highest nor lowest weight, and the spectrum of the elliptic operator is: $m = \alpha + n$, $n \in \mathbb{Z}$. We have flowed representations as well.

We start again from eq. (2.4). After extracting the poles corresponding to the discrete spectra, we are left with the following divergent expression:

$$
\begin{aligned}
Z_C = \frac{1}{\pi\eta^3\bar{\eta}^3}\frac{1}{2}\sum_{a,b\in\mathbb{Z}_2} P\int_{-\infty}^{\infty} ds (q\bar{q})^{\frac{s^2}{k}} \sum_{N,w_+,\tilde{f},\tilde{\tilde{f}}\in\mathbb{Z}} e^{i\pi(\tilde{f}-\tilde{\tilde{f}})b} \times \\
\times \sum_{r,\bar{r}\in\mathbb{Z}} \delta_{r-\bar{r}+\tilde{f}-\tilde{\tilde{f}},N} \Bigg\{ q^{\frac{1}{4(k+2)}\left(N-2\tilde{f}-a-kw_+\right)^2+\frac{k}{2(k+2)}\left(\tilde{f}+\frac{a}{2}+\frac{N-kw_+}{k}\right)^2} \times \\
\times\, \bar{q}^{\frac{1}{4(k+2)}\left(N+2\tilde{\tilde{f}}+a+kw_+\right)^2+\frac{k}{2(k+2)}\left(\tilde{\tilde{f}}+\frac{a}{2}+\frac{-N-kw_+}{k}\right)^2} \times \\
\times \frac{S_r\bar{S}_{\bar{r}}}{2is+1+r+\bar{r}+\tilde{f}+\tilde{\tilde{f}}+a+kw_+} - \\
- q^{\frac{1}{4(k+2)}\left(N-2(\tilde{f}+1)-a-k(w_++1)\right)^2+\frac{k}{2(k+2)}\left(\tilde{f}+1+\frac{a}{2}+\frac{N-k(w_++1)}{k}\right)^2+r} \times \\
\times\, \bar{q}^{\frac{1}{4(k+2)}(N+2(\tilde{\tilde{f}}+1)+a+k(w_++1))^2+\frac{k}{2(k+2)}\left(\tilde{\tilde{f}}+1+\frac{a}{2}+\frac{-N-k(w_++1)}{k}\right)^2+\bar{r}} \times \\
\times \frac{S_r\bar{S}_{\bar{r}}}{2is+1+r+\bar{r}+\tilde{f}+\tilde{\tilde{f}}+a+k(w_++1)} \Bigg\}.
\end{aligned}
\tag{3.19}
$$

Shifting in the second term $w_+ + 1 \to w_+$, $\tilde{f}+1\to\tilde{f}$, $\tilde{\tilde{f}}+1\to\tilde{\tilde{f}}$ (which is related to performing spectral flow by one unit in both the bosonic and the fermionic sector) and using the following identity:

$$
q^r S_r(\tau) = S_{-r}(\tau)\,, \tag{3.20}
$$

we obtain finally the expression:

$$
\begin{aligned}
Z_C = \frac{1}{\pi\eta^3\bar{\eta}^3}\frac{1}{2}\sum_{a,b\in\mathbb{Z}_2} P\int_{-\infty}^{\infty} ds (q\bar{q})^{\frac{s^2}{k}} \sum_{N,w_+,\tilde{f},\tilde{\tilde{f}}\in\mathbb{Z}} e^{i\pi(\tilde{f}-\tilde{\tilde{f}})b} q^{\frac{1}{4(k+2)}\left(N-2\tilde{f}-a-kw_+\right)^2} \times \\
\times q^{\frac{k}{2(k+2)}\left(\tilde{f}+\frac{a}{2}+\frac{N-kw_+}{k}\right)^2} \bar{q}^{\frac{1}{4(k+2)}\left(N+2\tilde{\tilde{f}}+a+kw_+\right)^2} \bar{q}^{\frac{k}{2(k+2)}\left(\tilde{\tilde{f}}+\frac{a}{2}+\frac{-N-kw_+}{k}\right)^2} \times \\
\times \sum_{r,\bar{r}\in\mathbb{Z}} \delta_{r-\bar{r}+\tilde{f}-\tilde{\tilde{f}},N} \left(S_r\bar{S}_{\bar{r}} - S_{-r-1}\bar{S}_{-\bar{r}-1}\right) \frac{1}{2is+1+r+\bar{r}+\tilde{f}+\tilde{\tilde{f}}+a+kw_+}\,.
\end{aligned}
\tag{3.21}
$$

Then, to proceed, we note that we have the following identity:

$$
S_r + S_{-r-1} = 1\,, \tag{3.22}
$$

that expresses the splitting of an $s=0$ (and $\alpha = 1/2$) continuous representation in two discrete representations. Using this expression we recast the last factor of eq. (3.21) in the following form:

$$
\sum_{r\in\mathbb{Z}} \left(S_r - \frac{1}{2}\right) \frac{1}{2is+1+2r+2\tilde{f}+a-N+kw_+} +
$$

JHEP06(2004)033

$$+\sum_{\bar r\in\mathbb{Z}}(\bar S_{\bar r}-1/2)\frac{1}{2is+1+2\bar r+2\tilde{\bar f}+a+N+kw_+}\,. \tag{3.23}$$

First we would like to isolate the divergent term from this expression. By expanding $\vartheta\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right](\tau,\nu)/\vartheta_1(\tau,\nu)$ in powers of $e^{2i\pi\nu}$, we can also prove the following generalized identity:

$$\sum_n(-)^{b(n+a/2)}q^{\frac12(n+a/2)^2}S_{N-n}+(-)^{ab}\sum_n(-)^{b(n+a/2)}q^{\frac12(n+a/2)^2}S_{-N-1-a-n}=\vartheta\begin{bmatrix}a\\b\end{bmatrix}(\tau,0)\,. \tag{3.24}$$

This allows to split naturally the left-moving and right-moving contributions of eq. (3.23) in sums over positive integers as follows:

$$\sum_{r=0}^{\infty}\left[\frac{\left(S_{r-\tilde f}-1\right)+1/2}{1+2r+2is+a-N+kw_+}+\frac{(-)^{ab}\left(S_{r-\tilde f-a}-1\right)+1/2}{1+2r-2is-a+N-kw_+}\right]+$$
$$+\sum_{\bar r=0}^{\infty}\left[\frac{\left(\bar S_{\bar r-\tilde{\bar f}}-1\right)+1/2}{1+2\bar r+2is+a+N+kw_+}+\frac{(-)^{ab}\left(\bar S_{\bar r-\tilde{\bar f}-a}-1\right)+1/2}{1+2\bar r-2is-a-N-kw_+}\right]. \tag{3.25}$$

One could have obtained naturally this decomposition by expanding directly the supersymmetric determinant $|\vartheta\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right]/\vartheta_1|^2$ in powers of the holonomies from the beginning of the computation. The first terms of the expansion of the function S_r, eq. (3.2) are:

$$S_r=1-e^{-2\pi\tau_2(r+1)}e^{2i\pi\tau_1(r+1)}+\mathcal{O}\left(e^{-2\pi\tau_2(2r+3)}\right).$$

Therefore, as $\tau_2>0$ in the fundamental domain, all the terms in the expansion of $S_{r-\tilde f}-1$ are sub-leading with respect to the constant $1/2$ term in the sum over $r\in\mathbb{N}$. This leading constant term gives a logarithmic divergence that needs to be correctly regularized. Since this term does not depend on the modular parameter τ it can be interpreted as a density of states.

The density of long string states. The procedure to regularize this divergent factor was already discussed in [32] for thermal AdS_3 and in a closer context in [14] for the bosonic coset. We adopt the same "Liouville wall" regularization as in the aforementioned papers and find contributions like:

$$\sum_{n=0}^{\infty}\frac{e^{-n\epsilon}}{A+n}=\log\epsilon+\frac{d}{dA}\log\Gamma(A)+\mathcal{O}(\epsilon)\,. \tag{3.26}$$

By using this regularization procedure for the four divergent pieces of (3.25) and adding the contributions of $s>0$ and $s<0$ from the integral, we obtain:

$$\begin{aligned}\rho(s;N,w_+;a)=&\ \frac{1}{\pi}\log\epsilon+ \qquad (3.27)\\ &+\frac{1}{8i\pi}\frac{d}{ds}\log\left\{\frac{\Gamma\left(\frac12-is+\frac a2-\frac N2+\frac k2w_+\right)\Gamma\left(\frac12-is-\frac a2+\frac N2-\frac k2w_+\right)}{\Gamma\left(\frac12+is+\frac a2-\frac N2+\frac k2w_+\right)\Gamma\left(\frac12+is-\frac a2+\frac N2-\frac k2w_+\right)}\times\right.\end{aligned}$$

$$\times \frac{\Gamma\left(\frac{1}{2}-is+\frac{a}{2}+\frac{N}{2}+\frac{k}{2}w_+\right)\Gamma\left(\frac{1}{2}-is-\frac{a}{2}-\frac{N}{2}-\frac{k}{2}w_+\right)}{\Gamma\left(\frac{1}{2}+is+\frac{a}{2}+\frac{N}{2}+\frac{k}{2}w_+\right)\Gamma\left(\frac{1}{2}+is-\frac{a}{2}-\frac{N}{2}-\frac{k}{2}w_+\right)}\Bigg\} .$$

The divergent factor in the first line has been argued in [32] to correspond to the divergence due to the infinite volume available for the long strings. The second one gives the regularized density of continuous representations. Thinking of the duality of the supersymmetric $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{U}(1)$ coset with $N=2$ Liouville, we can view this result in terms of the ratio of the reflection amplitudes in two $N=2$ Liouville theories differing by a flipping of sign in the exponential potential. In fact, our result matches the computations of two-point functions in N=2 Super-Liouville,[7] both in the NS and R sectors [33]. This is a method preserving $N=2$ superconformal symmetry to confine the strings to a finite volume and obtain a regularized density of states.[8] If we use the parameterization (3.10) for the the total current $\mathcal{J}^3$ and $\bar{\mathcal{J}}^3$ eigenvalues, this expression is just:

$$\begin{aligned}\rho(s;\tilde{m}_t,\tilde{\bar{m}}_t;a) &= \frac{1}{\pi}\log\epsilon+\frac{1}{8i\pi}\frac{d}{ds}\log\left\{\frac{\Gamma\left(\frac{1}{2}-is+\frac{a}{2}-\tilde{m}_t\right)\Gamma\left(\frac{1}{2}-is-\frac{a}{2}+\tilde{m}_t\right)}{\Gamma\left(\frac{1}{2}+is+\frac{a}{2}-\tilde{m}_t\right)\Gamma\left(\frac{1}{2}+is-\frac{a}{2}+\tilde{m}_t\right)}\times\right.\\ &\qquad\left.\times\frac{\Gamma\left(\frac{1}{2}-is+\frac{a}{2}-\tilde{\bar{m}}_t\right)\Gamma\left(\frac{1}{2}-is-\frac{a}{2}+\tilde{\bar{m}}_t\right)}{\Gamma\left(\frac{1}{2}+is+\frac{a}{2}-\tilde{\bar{m}}_t\right)\Gamma\left(\frac{1}{2}+is-\frac{a}{2}+\tilde{\bar{m}}_t\right)}\right\}\\ &= \frac{1}{\pi}\log\epsilon+\frac{1}{4i\pi}\frac{d}{ds}\log\left\{\frac{\Gamma\left(\frac{1}{2}-is+\frac{a}{2}-\tilde{m}_t\right)\Gamma\left(\frac{1}{2}-is-\frac{a}{2}+\tilde{\bar{m}}_t\right)}{\Gamma\left(\frac{1}{2}+is+\frac{a}{2}-\tilde{m}_t\right)\Gamma\left(\frac{1}{2}+is-\frac{a}{2}+\tilde{\bar{m}}_t\right)}\right\}\end{aligned} \tag{3.28}$$

where we have used in the last line the fact that $\tilde{m}_t-\tilde{\bar{m}}_t\in\mathbb{Z}$. To summarize, we have obtained the continuous spectrum in the following form:

$$Z_C=\frac{1}{2}\sum_{a,b\in\mathbb{Z}_2}\sum_{N,w_+\in\mathbb{Z}}P\int_0^\infty ds\rho(s;N,w_+;a)ch_c\left(s,\frac{N}{2}-\frac{k}{2}w_+\right)\begin{bmatrix}a\\b\end{bmatrix}\bar{ch}_c\left(s,-\frac{N}{2}-\frac{k}{2}w_+\right)\begin{bmatrix}a\\b\end{bmatrix}. \tag{3.29}$$

We have introduced the following continuous characters of the $N=2$ algebra (with $2m\in\mathbb{Z}$):

$$\begin{aligned}ch_c(s,m)\begin{bmatrix}a\\b\end{bmatrix}(\tau,\nu) &= \frac{1}{\eta^3}q^{\frac{s^2+m^2}{k}}z^{\frac{2m}{k}}\vartheta\begin{bmatrix}a\\b\end{bmatrix}(\tau,\nu)\\ &= \frac{1}{\eta}\sum_n e^{i\pi b\left(n+\frac{a}{2}\right)}\,z^{\frac{2m}{k}+n+\frac{a}{2}}q^{\frac{k}{2(k+2)}\left(n+\frac{a}{2}+\frac{2m}{k}\right)^2}\hat{\lambda}_{1/2+is,n+\frac{a}{2}-m}(\tau)\,.\end{aligned} \tag{3.30}$$

In the last line, we have decomposed the continuous $N=2$ characters in terms of the coset characters for the continuous representations:

$$\hat{\lambda}_{1/2+is,m}(\tau)=\frac{1}{\eta^2}q^{\frac{s^2}{k}+\frac{m^2}{k+2}}\,. \tag{3.31}$$

[7]These computations correspond to the sector $\tilde{m}_t=\tilde{\bar{m}}_t$.

[8]Note that the potential used to regularize the volume divergence is largely arbitrary and each choice of regularization would lead to a different regularized density of states. In our case, preservation of the extended $N=2$ superconformal symmetry severely limits the choice of regularization.

JHEP06(2004)033

Decomposition in extended continuous $N=2$ characters. We have already shown that the discrete characters of the supersymmetric coset can be decomposed into extended discrete characters of $N=2$. Now we would like to introduce the *extended continuous* characters of $N=2$ [12]. They are also given by (see [31] for details):

$$Ch_c(s,m)\begin{bmatrix}a\\b\end{bmatrix}(\tau,\nu) = \sum_{n\in\mathbb{Z}} ch_c(s,m+kn)\begin{bmatrix}a\\b\end{bmatrix}(\tau,\nu)$$
$$= q^{\frac{s^2}{k}}\frac{\vartheta\left[{a \atop b}\right](\tau,\nu)}{\eta(\tau)}\Theta_{2m,k}\left(\tau,\frac{2\nu}{k}\right) \tag{3.32}$$

where we have introduced the classical theta function at level k:

$$\Theta_{m,k}(\tau,\nu) = \sum_{n\in\mathbb{Z}} q^{k\left(n+\frac{m}{2k}\right)^2} z^{k\left(n+\frac{m}{2k}\right)}\,. \tag{3.33}$$

To decompose the partition function in terms of extended characters we need another regulator compatible with them. In fact, the density of continuous representations we obtained before, eq. (3.28), is not invariant under spectral flow shifts by k units. To obtain a invariant regulator, we could have started our computation with the expansion of the full determinant $|\vartheta\left[{a \atop b}\right]/\vartheta_1|^2\ \Theta_{m,k}\bar{\Theta}_{-m,k}$ in powers of the holonomies. Then the sum over r can be splitted using the the following identity:

$$\sum_{n,n'}(-)^{b(n+a/2)}q^{\frac{1}{2}(n+a/2)^2+k\left(n'+\frac{m}{2k}\right)^2}S_{N-n-kn'}+ \tag{3.34}$$
$$+(-)^{ab}\sum_{n,n'}(-)^{b(n+a/2)}q^{\frac{1}{2}(n+a/2)^2+k\left(n'-\frac{m}{2k}\right)^2}S_{-N-1-a-m-n-kn'} = \vartheta\begin{bmatrix}a\\b\end{bmatrix}(\tau,0)\Theta_{m,k}(\tau,0)\,.$$

Now we can split the variable N as for the discrete representations: $N=kp+e$, with $e\in\mathbb{Z}_k$, and then we define the mod $2k$ charge $2g=e+kt$, $t=0,1$. We obtain the following anti-symmetric combination of extended continuous $N=2$ characters:

$$Z_C = \frac{1}{2}\sum_{a,b\in\mathbb{Z}_2} P\int_0^\infty ds \sum_{2g\in\mathbb{Z}_{2k}}\rho(s,g,-g;a)Ch_c\left(s,g\right)\begin{bmatrix}a\\b\end{bmatrix}\times\bar{C}h_c\left(s,-g\right)\begin{bmatrix}a\\b\end{bmatrix} \tag{3.35}$$

with the following density of states:

$$\rho(s,g,\bar{g};a) = \frac{1}{\pi}\log\epsilon + \frac{1}{4i\pi}\frac{d}{ds}\log\left\{\frac{\Gamma\left(\frac{1}{2}-is+\frac{a}{2}-g\right)\Gamma\left(\frac{1}{2}-is-\frac{a}{2}+\bar{g}\right)}{\Gamma\left(\frac{1}{2}+is+\frac{a}{2}-g\right)\Gamma\left(\frac{1}{2}+is-\frac{a}{2}+\bar{g}\right)}\right\}. \tag{3.36}$$

Note that the leading divergent term is modular invariant by itself, as we now from [12] and [31]. We have learned here that different regularization procedures give different finite parts for the density of states. Finally we would like to stress that, due to the expression of the continuous characters, the leading contribution to the partition function is given by a free field computation.[9]

[9]We thank the referee for this remark.

3.3 Puzzling sectors of states

In the last subsections we have succeeded in decomposing the partition function in both discrete and continuous representations of the supersymmetric coset theory $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$. The contribution of the continuous representations came with an infrared divergence, that we regularized. The regularized expression gives a non-trivial density of states. However, the partition function contains also a piece corresponding to the the finite part of the partition function after the extraction of the discrete representations. These new sectors are explicitely given by:

$$\left[\sum_{r\in\mathbb{Z}} \frac{S_r}{\frac{1}{2}+is+\frac{a}{2}-\tilde{m}_t+r} - \sum_{r\in\mathbb{N}} \frac{1}{\frac{1}{2}+is+\frac{a}{2}-\tilde{m}_t+r}\right] \frac{q^{\frac{s^2+\tilde{m}_t^2}{k}}}{\eta^3}\vartheta\begin{bmatrix}a\\b\end{bmatrix}, \tag{3.37}$$

and we have a similar term with non-trivial contribution from the right-moving sector. These terms cannot be interpreted as coming from usual highest weight representations of $\mathrm{SL}(2,\mathbb{R})$. Our regularization procedure is not obviously modular invariant and/or compatible with the $\mathrm{SL}(2,\mathbb{R})$ affine symmetry of the parent theory. A modular invariant regularization is provided by using for the H_3 determinant the function: $1/|\vartheta_1(\nu|\tau)|^{2(1-\epsilon)}$, with $\epsilon>0$. However the explicit integration over the holonomies becomes challenging.

3.4 Comments on the Witten index

The twisted RR sector of the partition function: $\mathrm{Tr}_{RR}(-)^F q^{L_0}(-)^{\bar{F}}q^{\bar{L}_0}$ is a constant — the Witten index [22] — and we observe that it can be computed straightforwardly, using the lagrangian form of the un-decomposed partition function, eq. (2.7):

$$\begin{aligned} I_W &= k\int_0^1 ds_1\, ds_2 \frac{1}{\left|\vartheta\left[\begin{smallmatrix}1+2s_1\\1+2s_2\end{smallmatrix}\right]\right|^2} \sum_{M,w_+\in\mathbb{Z}} e^{-\frac{k\pi}{\tau_2}|(w_++s_1)\tau-(M+s_2)|^2} \left|\vartheta\begin{bmatrix}1+2s_1\\1+2s_2\end{bmatrix}\right|^2 \\ &= k\int_{\mathbb{R}^2} du_1\, du_2\, e^{-\frac{k\pi}{\tau_2}|u_1\tau-u_2|^2} = 1\,. \end{aligned} \tag{3.38}$$

Even if the partition function for the super-coset, eq. (2.4) is divergent, as the bosonic coset partition function [14], the Witten index is finite. However the computation is very sensitive to the normalization of the path integral that lead to the expression (2.1). We will explain now how the present computation is consistent with the decomposition of the supersymmetric coset partition function into $N=2$ characters.

Contribution of the extended discrete characters: we recall that the extended discrete characters of $N=2$ are given by:

$$Ch_d(j,r)\begin{bmatrix}a\\b\end{bmatrix}(\tau,\nu) = \frac{1}{\eta^3(\tau)}\sum_{n\in\mathbb{Z}} q^{\frac{-(j-1/2)^2+(j+r+a/2+kn)^2}{k}} z^{\frac{2j+2r+a}{k}} \frac{\vartheta\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right](\tau,\nu)}{1+(-)^b z\, q^{(1+a)/2+r+kn}}\,. \tag{3.39}$$

Therefore in the twisted Ramond sector we have:

$$Ch_d(j,r)\begin{bmatrix}1\\1\end{bmatrix}(\tau,0) = \sum_{n\in\mathbb{Z}} q^{\frac{-(j-1/2)^2+(j+r+1/2+kn)^2}{k}} \delta_{r+kn+1,0} = \delta_{r+1,0 \mod k}\,. \tag{3.40}$$

So the full contribution of the discrete part of the partition function to the index is:

$$\begin{aligned} I_D &= \sum_{2j=1}^{k+1} \Upsilon(2j,1) \sum_{m\in\mathbb{Z}_{2k}} Ch_d\left(j,\frac{m}{2}-j\right)\begin{bmatrix}1\\1\end{bmatrix}\bar{Ch}_d\left(j,-\frac{m}{2}-j\right)\begin{bmatrix}1\\1\end{bmatrix} \\ &= \sum_{2j=1}^{k+1} \Upsilon(2j,1) \sum_{m\in\mathbb{Z}_{2k}} \delta_{m-2j,0 \mod 2} \times \delta_{m-2j+2,0 \mod 2k} \times \delta_{m+2j-2,0 \mod 2k} \\ &= 1 \end{aligned} \tag{3.41}$$

Therefore only one primary state from the discrete representations: $j=1$, $m=0$ (left and right) contribute to the Witten index in the axial coset.

The continuous representations — and the puzzling sectors as well — do not contribute to the Witten index. So the result is in perfect agreement with the direct computation of the Witten index using the lagrangian partition function.

4. The double scaled little string theory

The little string theory [34, 35] (LST) is the decoupling limit of the theory living on the NS five-branes, obtained by sending the string coupling constant to zero. This theory is non-gravitational but shares many properties with conventional string theories. An holographic description is provided by the near-horizon geometry of the five-branes background, which is an exact superconformal field theory [36, 37, 38] based on (for k five-branes) $\mathrm{SU}(2)_k \times \mathbb{R}_{Q^2=2/k}$. However, because of the linear dilaton this string background is strongly coupled down the throat. The authors of [9] show that this strong coupling problem can be avoided by going to the Higgs phase of the little string theory, and taking a double scaling limit. Using the duality between $N=2$ super-Liouville and the supersymmetric coset $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$, they argue that this background (the DSLST) is given by the following orbifold:

$$\frac{\mathrm{SL}(2,\mathbb{R})_k/\mathrm{U}(1) \times \mathrm{SU}(2)_k/\mathrm{U}(1)}{\mathbb{Z}_k}.$$

This worldsheet CFT also describes superstrings on a singular non-compact Calabi-Yau space, in this case on an A_{k-1} ALE manifold. [39] Another regularization of the strong coupling problem is given by a null deformation of AdS_3, corresponding to adding a stack of fundamental strings to the five-branes setup [17]. In the following we will compute the partition function for the superconformal field theory of the double scaled little string theory, and show explicitely that it gives a spacetime supersymmetric string background (related issues were considered in [40]).

4.1 $N=2$ minimal models: a reminder

Here we would like to recall the partition function of the supersymmetric $\mathrm{SU}(2)_k/\mathrm{U}(1)$ coset. The building blocks are the N=2 minimal models characters which are given by (see [41, 42, 43]):

$$\chi_m^{j(s)}(\tau,\nu) = \sum_{n\in\mathbb{Z}_{k-2}} c^j_{m+4n-s}(\tau)\Theta_{2m+k(4n-s),2k(k-2)}\left(\tau,\frac{\nu}{k}\right) \tag{4.1}$$

where the c^j_m are the $\mathrm{SU}(2)_{k-2}$ string functions. The charge s is defined modulo 4: $s = 0, 2$ are the NS sectors with $(-)^F = +1, -1$ and $s = 1, 3$ are the R sectors with $(-)^F = +1, -1$. Note that in the NS sector, we have $2j + m$ even, while in the R sector, $2j + m$ is odd. We have the following symmetry:

$$\chi_{m+k}^{(k-2)/2-j\,(s+2)}(\tau,\nu) = \chi_m^{j\,(s)}(\tau,\nu)\,, \tag{4.2}$$

and the modular transformation:

$$\chi_m^{j\,(s)}\left(-\frac{1}{\tau},0\right) = \frac{1}{k}\sum_{j',m',s'} \sin\left(\pi\frac{(2j+1)(2j'+1)}{k}\right) e^{i\pi\left(\frac{mm'}{k}-\frac{ss'}{2}\right)}\chi_{m'}^{j'\,(s')}(\tau,0)\,. \tag{4.3}$$

Grouping the NS and R states we obtain the RNS characters:

$$\mathcal{C}^j_m\begin{bmatrix}a\\b\end{bmatrix} = \left(\chi_m^{j\,(a)} + (-)^b\chi_m^{j\,(a+2)}\right) e^{i\pi\frac{ab}{2}} \tag{4.4}$$

with the symmetry:

$$\mathcal{C}^{(k-2)/2-j}_{m+k}\begin{bmatrix}a\\b\end{bmatrix} = (-)^b\mathcal{C}^j_m\begin{bmatrix}a\\b\end{bmatrix}. \tag{4.5}$$

Then a type 0B modular invariant is given by:

$$Z_{MM} = \frac{1}{2}\sum_{a,b\in\mathbb{Z}_2}\sum_{2j,2j'=0}^{k-2}\sum_{n,n'\in\mathbb{Z}_{2k}} N_{jj'}M^{nn'}\mathcal{C}^j_n\begin{bmatrix}a\\b\end{bmatrix}\bar{\mathcal{C}}^{j'}_{n'}\begin{bmatrix}a\\b\end{bmatrix} \tag{4.6}$$

where $N_{jj'}$ is any modular invariant of SU(2), for example the A-invariant: $N_{jj'} = \delta_{j,j'}$.

For the modular invariant of the level k theta functions $M^{mm'}$ we can choose the diagonal one $\delta_{m,m'}$. Of great importance for the following will be the $\mathbb{Z}_k$ orbifold modular invariant:

$$Z = \sum_{\substack{n-n'=0 \mod 2k\\ n+n'=0 \mod 2}} \Theta_{n,k}\bar{\Theta}_{n',k} = \frac{1}{k}\sum_{\gamma',\delta'\in\mathbb{Z}_k}\sum_{n\in\mathbb{Z}_{2k}} e^{2i\pi(n-\gamma')\frac{\delta'}{k}}\Theta_{n,k}\Theta_{-n+2\gamma',k}\,.$$

Then we have the following orbifold sectors:

$$Z_{MM}\begin{bmatrix}a;a\\b;b\end{bmatrix}\begin{bmatrix}\gamma'\\\delta'\end{bmatrix} = \sum_{2j=0}^{k-2}\sum_{m\in\mathbb{Z}_{2k}}\mathcal{C}^j_m\begin{bmatrix}a\\b\end{bmatrix}\bar{\mathcal{C}}^{j'}_{-m+2\gamma'}\begin{bmatrix}a\\b\end{bmatrix}e^{2i\pi(m-\gamma')\frac{\delta'}{k}}\,. \tag{4.7}$$

4.2 Combining the two cosets

From now on we consider the two coset theories together, $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{U}(1)$ and $\mathrm{SU}(2)/\,\mathrm{U}(1)$ with the same supersymmetric level k. The central charges add up to $c = 6$:

$$c = c_{\mathrm{SL}(2,\mathbb{R})/\,\mathrm{U}(1),\mathrm{susy}} + c_{\mathrm{SU}(2)/\,\mathrm{U}(1),\mathrm{susy}} = \frac{3(k+2)}{k} + \frac{3(k-2)}{k} = 6\,. \tag{4.8}$$

In order to obtain eventually a spacetime supersymmetric background, we have to consider a $\mathbb{Z}_k$ orbifold of this superconformal field theory.

$N = 2$ charges. Let us begin with the $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{U}(1)$ part. The N=2 charges of the twisted 2D black hole are, from eq. (2.9):

$$\mathcal{Q}_L = \tilde{f} + \frac{a}{2} + \frac{N-\gamma-kw_+}{k}\,, \qquad \mathcal{Q}_R = \tilde{\bar{f}} + \frac{a}{2} - \frac{N+\gamma+kw_+}{k}\,, \tag{4.9}$$

combined with a projector: $e^{2i\pi N\delta/k}$.

Now let us consider the $\mathrm{SU}(2)/\,\mathrm{U}(1)$ factor. From eq. (4.1), The $N = 2$ charges of the twisted $N = 2$ minimal model are:

$$\mathcal{Q}'_L = -\frac{s}{2} + \frac{m}{k} \mod 2\,, \qquad \mathcal{Q}'_R = -\frac{s}{2} + \frac{-m+2\gamma'}{k} \mod 2 \tag{4.10}$$

combined with a projector: $e^{2i\pi(m-\gamma')\delta'/k}$.

The $\mathbb{Z}_k$ orbifold. If we choose to identify the two projections (i.e. we mod out by a diagonal $\mathbb{Z}_k$), namely $\gamma' = \gamma$, $\delta' = \delta$, we find the following set of total N=2 charges:

$$\begin{aligned}\mathcal{Q}_L^{\mathrm{tot}} &= \tilde{f} + \frac{a}{2} - \frac{s}{2} + \frac{m+N-\gamma}{k} - w_+ \mod 2\\ \mathcal{Q}_R^{\mathrm{tot}} &= \tilde{\bar{f}} + \frac{a}{2} - \frac{s}{2} - \frac{N+m-\gamma}{k} - w_+ \mod 2\end{aligned} \tag{4.11}$$

with the projector:

$$e^{2i\pi(m-\gamma+N)\frac{\delta}{k}}.$$

Summing over δ gives the constraint

$$m - \gamma + N = 0 \mod k := kp\,, \tag{4.12}$$

such that the left and right $N = 2$ charges are both integer. So the type 0B partition function for this theory is:

$$Z^{(0B)}_{\mathrm{SL}(2,\mathbb{R})/\,\mathrm{U}(1)\times\mathrm{SU}(2)/\,\mathrm{U}(1)} = \frac{1}{2}\sum_{a,b\in\mathbb{Z}_2}\frac{1}{2k}\sum_{\gamma,\delta\in\mathbb{Z}_k} Z_{BH}\begin{bmatrix}a;a\\b;b\end{bmatrix}\begin{bmatrix}\gamma\\\delta\end{bmatrix} Z_{MM}\begin{bmatrix}a;a\\b;b\end{bmatrix}\begin{bmatrix}\gamma\\\delta\end{bmatrix}. \tag{4.13}$$

Then we conclude that the $\mathbb{Z}_k$ orbifold of the superconformal theory $\mathrm{SL}(2,\mathbb{R})_k/\,\mathrm{U}(1)\times\mathrm{SU}(2)_k/\,\mathrm{U}(1)$ has an integer spectrum of $N = 2$ R-charges. It will then be possible to construct a spacetime supersymmetric vacua using the spectral flow of the $N = 2$ algebra [44]. Now we consider the complete superstring background for the doubly scaled little string theory, i.e. the orbifold of $\mathrm{SL}(2,\mathbb{R})_k/\,\mathrm{U}(1)\times\mathrm{SU}(2)_k/\,\mathrm{U}(1)$ times six dimensional Minkowski space. In the light-cone gauge we obtain the following partition function for type $0B$ DSLST:

$$Z^{(0B)}_{DSLST} = \frac{1}{2}\sum_{a,b\in\mathbb{Z}_2}\frac{1}{2k}\sum_{\gamma,\delta\in\mathbb{Z}_k} Z_{BH}\begin{bmatrix}a;a\\b;b\end{bmatrix}\begin{bmatrix}\gamma\\\delta\end{bmatrix} Z_{MM}\begin{bmatrix}a;a\\b;b\end{bmatrix}\begin{bmatrix}\gamma\\\delta\end{bmatrix}\frac{1}{(8\pi^2\tau_2)^2\eta^4\bar{\eta}^4}\frac{\vartheta^2\left[{a\atop b}\right]}{\eta^2}\frac{\bar{\vartheta}^2\left[{a\atop b}\right]}{\bar{\eta}^2}\,. \tag{4.14}$$

4.3 $N = 4$ structure of the double scaled little string theory

In this section we would like to study the underlying $N = 4$ structure of the theory, and in particular relate the characters we obtained to the $N = 4$ characters of the near-horizon geometry of coinciding NS5-branes — the $\mathrm{SU}(2)_k\times\mathbb{R}_Q$ SCFT.

The $N=4$ algebra. The superconformal symmetry of the theory $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)\times\mathrm{SU}(2)/\mathrm{U}(1)$ is enlarged to a small N=4 superconformal algebra, due to the $\mathbb{Z}_k$ orbifold. Explicitely, we have the following supercurrents and $\mathrm{SU}(2)_1$ currents [45]:

$$\begin{aligned}
G^{\pm} &= -\sqrt{\frac{2}{k}}\left[\sqrt{k+2}\,\pi^{-}e^{-i\sqrt{\frac{k-2}{2(k+2)}}H^{-}} + \sqrt{k-2}\,\psi^{-}e^{i\sqrt{\frac{k+2}{2(k-2)}}H^{-}}\right]e^{\pm\frac{i}{\sqrt{2}}H^{+}}\\
\tilde{G}^{\pm} &= \sqrt{\frac{2}{k}}\left[\sqrt{k+2}\,\pi^{+}e^{i\sqrt{\frac{k-2}{2(k+2)}}H^{-}} - \sqrt{k-2}\psi^{+}e^{-i\sqrt{\frac{k+2}{2(k-2)}}H^{-}}\right]e^{\pm\frac{i}{\sqrt{2}}H^{+}}\\
(S^3,S^{\pm}) &= \left(\frac{i}{\sqrt{2}}\partial H^{+}, e^{\pm i\sqrt{2}H^{+}}\right). \qquad (4.15)
\end{aligned}$$

Here $\psi^{\pm}$ are the parafermionic currents of $\mathrm{SU}(2)/\mathrm{U}(1)$, $\pi^{\pm}$ are those of $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ and $H^{\pm}$ are free bosons. The boson H^{+} is compactified at the self-dual radius $R=\sqrt{2}$ — in order to have an $\mathrm{SU}(2)_1$ R-symmetry — and the boson H^{-} is compactified at the radius:

$$R_{H_-} = \sqrt{\frac{2(k+2)}{k-2}}\,.$$

To exhibit the $N=4$ structure in our partition function, eq. (4.13), we use the identity:

$$\frac{k}{2(k+2)}\mathcal{Q}^2 + \frac{k}{2(k-2)}\mathcal{Q}'^2 = \frac{1}{4}\left(\mathcal{Q}+\mathcal{Q}'\right)^2 + \frac{k-2}{4(k+2)}\left(\mathcal{Q}-\frac{k+2}{k-2}\mathcal{Q}'\right)^2. \qquad (4.16)$$

After the $\mathbb{Z}_k$ orbifold, the charges of the first factor are half-integer as they should be for an $\mathrm{SU}(2)_1$ lattice.

On the equivalence between DSLST and LST superconformal algebras. We rewrite the generators of the $N=4$ algebra for $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)\times\mathrm{SU}(2)/\mathrm{U}(1)$, eq. (4.15), with a free field representation for the $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ piece [28]:

$$\begin{aligned}
G^{\pm} &= -\sqrt{\frac{2}{k}}\left[\left(\sqrt{\frac{k+2}{2}}\partial X - i\sqrt{\frac{k}{2}}\partial\rho\right)e^{-i\sqrt{\frac{2}{k+2}}X - i\sqrt{\frac{k-2}{2(k+2)}}H^{-}} + \sqrt{k-2}\,\psi^{-}e^{i\sqrt{\frac{k+2}{2(k-2)}}H^{-}}\right]\times\\
&\times\quad e^{\pm\frac{i}{\sqrt{2}}H^{+}}, \qquad (4.17)
\end{aligned}$$

and we have a similar expression for the other SU(2) doublet. The fields $X(z)$ and $\rho(z)$ are canonically normalized chiral bosons, with a background charge on ρ: $Q_\rho=\sqrt{2/k}$. Now we perform the following $O(2)$ rotation between the free fields X and H^{-}:

$$\begin{aligned}
H^{-} &= \cos\theta\,\hat{H}^{-} - \sin\theta\,\hat{X}\\
X &= \cos\theta\,\hat{X} + \sin\theta\,\hat{H}^{-}\,, \qquad \text{with } \cos^2\theta = \frac{k-2}{k+2}. \qquad (4.18)
\end{aligned}$$

And one obtains the following supercurrents:

$$\begin{aligned}
\hat{G}^{\pm} &= -\sqrt{\frac{2}{k}}\left[\left(\sqrt{\frac{k-2}{2}}\partial\hat{X} + \sqrt{2}\partial\hat{H}^{-} - i\sqrt{\frac{k}{2}}\partial\rho\right)e^{-\frac{i}{\sqrt{2}}\hat{H}^{-}} + \sqrt{k-2}\,\psi^{-}e^{-i\sqrt{\frac{2}{k-2}}\hat{X}+\frac{i}{\sqrt{2}}\hat{H}^{-}}\right]\times\\
&\times\quad e^{\pm\frac{i}{\sqrt{2}}H^{+}}, \qquad (4.19)
\end{aligned}$$

Now, if we introduce the purely bosonic current of the $\mathrm{SU}(2)_{k-2}$ Cartan sub-algebra:

$$I^3(z) = i\sqrt{\frac{k-2}{2}}\partial\hat{X} \ , \tag{4.20}$$

One obtains exactly the $N=4$ algebra for $\mathrm{SU}(2)_k \times \mathbb{R}_{Q^2=2/k}$ [45], i.e. the near-horizon limit of coincident five-branes background. The theory differs only by the $N=2$ super-Liouville potential, that breaks the SO(4) R-symmetry of the LST (the $\mathrm{SU}(2)_{k-1} \times \mathrm{SU}(2)_{k-1}$ isospin of the N=4 algebra) to the $\mathrm{U}(1) \times \mathbb{Z}_k$ symmetry of the DSLST.

Matching the continuous spectra. Implementing this isomorphism, we expect that the continuous spectrum, that correspond to asymptotic states, will be the same in $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{U}(1) \times \mathrm{SU}(2)/\,\mathrm{U}(1)$ — the doubly scaled LST — and in $\mathrm{SU}(2) \times \mathbb{R}_Q$ — the LST. Explicitely we have for the $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{U}(1) \times \mathrm{SU}(2)/\,\mathrm{U}(1)$ (still in type 0B):

$$\begin{aligned} Z_C = \frac{1}{4k} \sum_{a,b\in\mathbb{Z}_2} \sum_{\gamma,\delta\in\mathbb{Z}_k} P\int_0^\infty ds \sum_{2g\in\mathbb{Z}_{2k}} e^{4i\pi g\frac{\delta}{k}} \rho(s,g;\gamma;a) Ch_c\left(s, g-\gamma/2\right) \begin{bmatrix} a \\ b \end{bmatrix} \times \\ \times \bar{C}h_c\left(s,-g-\gamma/2\right) \begin{bmatrix} a \\ b \end{bmatrix} \times \sum_{2j=0}^{k-2} \sum_{m\in\mathbb{Z}_{2k}} \mathcal{C}^j_m \begin{bmatrix} a \\ b \end{bmatrix} \bar{\mathcal{C}}^j_{-m+2\gamma} \begin{bmatrix} a \\ b \end{bmatrix} e^{2i\pi(m-\gamma)\frac{\delta}{k}} \, . \end{aligned} \tag{4.21}$$

With the density of states:

$$\rho(s,g;\gamma;a) = \frac{1}{\pi}\log\epsilon + \frac{1}{4i\pi}\frac{d}{ds}\log\left\{ \frac{\Gamma\left(\frac{1}{2}-is+\frac{a}{2}-g+\frac{\gamma}{2}\right)\Gamma\left(\frac{1}{2}-is-\frac{a}{2}-g-\frac{\gamma}{2}\right)}{\Gamma\left(\frac{1}{2}+is+\frac{a}{2}-g+\frac{\gamma}{2}\right)\Gamma\left(\frac{1}{2}+is-\frac{a}{2}+g-\frac{\gamma}{2}\right)} \right\} . \tag{4.22}$$

We should only compare the leading divergent part of the density of states to match the spectra of the two theories, because the regularized density depends on the phase shift of the $N=2$ super-Liouville potential which we used to regularize the partition function. For the divergent part, we obtain then, after solving for the constraints:

$$\begin{aligned} \frac{\log\epsilon}{4k\pi} \sum_{a,b\in\mathbb{Z}_2} \sum_{\gamma,\delta\in\mathbb{Z}_k} \sum_{2j=0}^{k-2} \int_0^\infty ds \sum_{m\in\mathbb{Z}_{2k}} \Bigg\{ \left(Ch_c\left(s,-\frac{m}{2}\right) \begin{bmatrix} a \\ b \end{bmatrix} \times \mathcal{C}^j_m \begin{bmatrix} a \\ b \end{bmatrix} \right) \times \\ \times \left(\bar{C}h_c\left(s,-\frac{m+2\gamma}{2}\right) \begin{bmatrix} a \\ b \end{bmatrix} \times \bar{\mathcal{C}}^j_{-m+2\gamma} \begin{bmatrix} a \\ b \end{bmatrix} \right) + \\ + \left(Ch_c\left(s,-\frac{m}{2}\right) \begin{bmatrix} a \\ b \end{bmatrix} \times \mathcal{C}^j_{m+k} \begin{bmatrix} a \\ b \end{bmatrix} \right) \times \\ \times \left(\bar{C}h_c\left(s,-\frac{m+2\gamma}{2}\right) \begin{bmatrix} a \\ b \end{bmatrix} \times \bar{\mathcal{C}}^j_{-m+2\gamma+k} \begin{bmatrix} a \\ b \end{bmatrix} \right) \Bigg\} . \end{aligned} \tag{4.23}$$

Now we use the decomposition of extended continuous characters, eq. (3.32), and obtain:

$$\begin{aligned} \frac{\log\epsilon}{4k\pi} \sum_{a,b\in\mathbb{Z}_2} \sum_{\gamma,\delta\in\mathbb{Z}_k} \sum_{2j=0}^{k-2} \int_0^\infty ds \sum_{m\in\mathbb{Z}_{2k}} \Bigg\{ \left(\frac{q^{\frac{s^2}{k}}}{\eta^3} \Theta_{-m,k} \vartheta \begin{bmatrix} a \\ b \end{bmatrix} \times \mathcal{C}^j_m \begin{bmatrix} a \\ b \end{bmatrix} \right) \times \\ \times \left(\frac{\bar{q}^{\frac{s^2}{k}}}{\bar{\eta}^3} \bar{\Theta}_{-m+2\gamma,k} \bar{\vartheta} \begin{bmatrix} a \\ b \end{bmatrix} \times \mathcal{C}^j_{-m+2\gamma} \begin{bmatrix} a \\ b \end{bmatrix} \right) + \end{aligned}$$

$$+\left(\frac{q^{\frac{s^2}{k}}}{\eta^3}\Theta_{-m,k}\vartheta\begin{bmatrix}a\\b\end{bmatrix}\times\mathcal{C}^j_{m+k}\begin{bmatrix}a\\b\end{bmatrix}\right)\times$$

$$\times\left(\frac{\bar{q}^{\frac{s^2}{k}}}{\bar{\eta}^3}\bar{\Theta}_{-m+2\gamma,k}\bar{\vartheta}\begin{bmatrix}a\\b\end{bmatrix}\times\mathcal{C}^j_{-m+2\gamma+k}\begin{bmatrix}a\\b\end{bmatrix}\right)\Bigg\}\,. \quad (4.24)$$

Using the decomposition of SU(2) characters in terms of characters of the $N=2$ minimal models:

$$\chi^j(\tau)\vartheta\begin{bmatrix}a\\b\end{bmatrix}(\tau,\nu)=\sum_{m\in\mathbb{Z}_{2k}}\mathcal{C}^j_m\begin{bmatrix}a\\b\end{bmatrix}(\tau,\nu)\Theta_{m,k}\left(\tau,-\frac{2\nu}{k}\right), \quad (4.25)$$

and the charge conjugation symmetry of the $N=2$ minimal model characters, eq (4.5), we get finally (see also [46]):

$$\frac{\log\epsilon}{2\pi}\sum_{a,b\in\mathbb{Z}_2}\sum_{2j=0}^{k-2}\int_0^\infty ds(q\bar{q})^{\frac{s^2}{k}}\chi^j\bar{\chi}^j\vartheta^2\begin{bmatrix}a\\b\end{bmatrix}\bar{\vartheta}^2\begin{bmatrix}a\\b\end{bmatrix}. \quad (4.26)$$

Putting the extra factors, one obtain for the full DSLST background:

$$\frac{\log\epsilon}{2\pi}\sum_{a,b\in\mathbb{Z}_2}\sum_{2j=0}^{k-2}\int_0^\infty ds\ (q\bar{q})^{\frac{s^2}{k}}\chi^j\bar{\chi}^j\vartheta^2\begin{bmatrix}a\\b\end{bmatrix}\bar{\vartheta}^2\begin{bmatrix}a\\b\end{bmatrix}\frac{1}{(8\pi^2\tau_2)^2\eta^4\bar{\eta}^4}\frac{\vartheta^2\left[{a\atop b}\right]}{\eta^2}\frac{\bar{\vartheta}^2\left[{a\atop b}\right]}{\bar{\eta}^2}\,. \quad (4.27)$$

This is actually the type $0B$ partition function for the little string theory [38], i.e. the coincident five-branes background. This result is intuitive. In fact, the spectra of the asymptotic states -continuous representations - match because the asymptotic geometry is insensitive to the fact that the five-branes are distributed on a circle in the transverse space.

4.4 Spacetime supersymmetry

In the previous section we have combined the $SL(2,\mathbb{R})/U(1)$ and the $SU(2)/U(1)$ supersymmetric coset with a $\mathbb{Z}_k$ orbifold, such that the $N=2$ charges are all integer, or, equivalently, such that the states of the theory belongs to representations of the small $N=4$ algebra. A spacetime-supersymmetric theory is obtained from a type 0 theory by twisting with the operator $(-)^{F_L}$ acting on the left worldsheet fermion number. Let us consider separately the continuous and discrete part of the spectrum.[10]

Continuous representations. Starting with continuous representations of $SL(2,\mathbb{R})/U(1)$, we get the following piece of the partition function after the twisting:

$$Z^C_{DLST}=\frac{1}{2}\sum_{a,b\in\mathbb{Z}_2}(-)^{a+b}\frac{1}{2}\sum_{\bar{a},\bar{b}\mathbb{Z}_2}(-)^{\bar{a}+\bar{b}+\eta\bar{a}\bar{b}}\sum_{2j=0}^{k-2}P\int_0^\infty ds\sum_{m,\bar{m}\in\mathbb{Z}_{2k}}\rho(s;m,\bar{m};a,\bar{a})\times$$

[10]In this analysis we don't consider the case of the new sectors of states since we don't have a good explanation for them — however the twisting of the theory can be done in the same way.

$$\times \left(Ch_c\left(s,-\frac{m}{2}\right)\begin{bmatrix}a\\b\end{bmatrix} \times \mathcal{C}^j_m \begin{bmatrix}a\\b\end{bmatrix}\right) \left(\bar{C}h_c\left(s,-\frac{\bar{m}}{2}\right)\begin{bmatrix}\bar{a}\\\bar{b}\end{bmatrix} \times \mathcal{C}^j_{\bar{m}} \begin{bmatrix}\bar{a}\\\bar{b}\end{bmatrix}\right) \times$$
$$\times \frac{1}{(8\pi^2\tau_2)^2\eta^4\bar{\eta}^4} \frac{\vartheta^2\left[{a \atop b}\right]}{\eta^2} \frac{\bar{\vartheta}^2\left[{\bar{a} \atop \bar{b}}\right]}{\bar{\eta}^2}\,. \tag{4.28}$$

In this expression, $\eta = 0$ for type IIB superstrings and $\eta = 1$ for type IIA superstrings. As we have already noticed in the type 0B partition function, the leading divergent part of the density of states gives the partition function for the coincident NS5-brane background.

Discrete representations. We now consider the piece coming from the discrete representations of $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{U}(1)$. In the same way, we obtain the following expression:

$$Z^D_{DLST} = \frac{1}{2}\sum_{a,b\in\mathbb{Z}_2}(-)^{a+b}\frac{1}{2}\sum_{\bar{a},\bar{b}\mathbb{Z}_2}(-)^{\bar{a}+\bar{b}+\eta\bar{a}\bar{b}}\sum_{2j=0}^{k-2}\sum_{2j'=1}^{k+1}\Upsilon(2j',1)\times$$
$$\times \sum_{m,\bar{m}\in\mathbb{Z}_{2k}} \left(Ch_d\left(j',-\frac{m}{2}-j'\right)\begin{bmatrix}a\\b\end{bmatrix} \times \mathcal{C}^j_m \begin{bmatrix}a\\b\end{bmatrix}\right)\times$$
$$\times \left(\bar{C}h_d\left(j',-\frac{\bar{m}}{2}-j'\right)\begin{bmatrix}\bar{a}\\\bar{b}\end{bmatrix} \times \mathcal{C}^j_{\bar{m}} \begin{bmatrix}\bar{a}\\\bar{b}\end{bmatrix}\right) \frac{1}{(8\pi^2\tau_2)^2\eta^4\bar{\eta}^4} \frac{\vartheta^2\left[{a \atop b}\right]}{\eta^2} \frac{\bar{\vartheta}^2\left[{\bar{a} \atop \bar{b}}\right]}{\bar{\eta}^2}\,. \tag{4.29}$$

From this discrete spectrum we can compute the Witten index of the theory, that counts the number of Ramond ground states, for the $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{U}(1) \times \mathrm{SU}(2)/\,\mathrm{U}(1)$ part of the background. We recall that the contribution to the Witten index of, respectively, the discrete extended characters of the supersymmetric $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{U}(1)$ and the $N=2$ minimal model characters are:

$$Ch_d(j',r)\begin{bmatrix}1\\1\end{bmatrix}(\tau,0) = \delta_{r+1,0 \mod k} \qquad \text{and} \qquad \mathcal{C}^j_m\begin{bmatrix}1\\1\end{bmatrix}(\tau,0) = \delta_{2j,m}\,. \tag{4.30}$$

Therefore, the Witten index for the DSLST is:

$$\sum_{2j=0}^{k-2}\sum_{2j'=1}^{k+1}\Upsilon(2j',1 \mod k)\sum_{m,\bar{m}\in\mathbb{Z}_{2k}}\delta_{2j,m}\delta_{2j,\bar{m}}\times\delta_{m-2j'+2,0}\delta_{\bar{m}-2j'+2,0} = k-1\,. \tag{4.31}$$

Note that our result is in agreement with [40]. It matches the cohomology of the A_{k-1} ALE space, which is dual to this worldsheet CFT.

5. The NS5-branes on a circle and the exact coset CFT

In this section we would like to match the holographic description of the little string theory in a double scaling limit — the orbifold of $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{U}(1) \times \mathrm{SU}(2)/\,\mathrm{U}(1)$ discussed in the last section — and the concrete supergravity description of the Higgs phase of little string theory, i.e. a distribution of NS5-branes in the transverse space.

5.1 The supergravity solution

Following [13], the supergravity for k NS5 branes spread over a circle in the transverse space with $\mathbb{Z}_k$ symmetry is (in the string frame):[11]

$$ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu + H(\rho,\psi)(dr^2 + r^2 d\phi^2 + d\rho^2 + \rho^2 d\psi^2) \tag{5.1}$$

where the conformal factor is given by the following sum with $\mathbb{Z}_k$ symmetry:

$$\begin{aligned} H &= 1 + \sum_{n=0}^{k-1} \frac{\alpha'}{r^2 + \rho^2 + \rho_0^2 - 2\rho_0\rho\cos(2\pi n/k - \psi)} \\ &= 1 + \frac{\alpha' k}{2\rho_0\rho\sinh x}\Lambda_k(x,\psi) \end{aligned} \tag{5.2}$$

with:

$$e^x = \frac{r^2+\rho^2+\rho_0^2}{2\rho_0\rho} + \sqrt{\left(\frac{r^2+\rho^2+\rho_0^2}{2\rho\rho_0}\right)^2 - 1} \tag{5.3}$$

and

$$\Lambda_k(x,\psi) = \frac{\sinh kx}{\cosh kx - \cos k\psi}\,. \tag{5.4}$$

This function captures the discreteness of the distribution of five-branes with $\mathbb{Z}_k$ symmetry. In the large k limit, $\Lambda_k(x,\psi) \to 1$, and we obtain a continuous distribution of NS5-branes. We now take the double scaling limit:

$$g_s \to 0\,, \qquad \alpha'\text{fixed}\,, \qquad U_0 \equiv \frac{\rho_0}{g_s\alpha'}\text{fixed}\,, \qquad u_i \equiv \frac{x^i}{g_s\alpha'}\text{fixed}\,. \tag{5.5}$$

Then, defining the coordinates:

$$\begin{aligned} u_1 &= \rho_0 \sinh\rho\cos\theta\cos\varphi\,, \qquad & u_2 &= \rho_0\sinh\rho\cos\theta\sin\varphi \\ u_3 &= \rho_0\cosh\rho\sin\theta\cos\psi\,, \qquad & u_4 &= \rho_0\cosh\rho\sin\theta\sin\psi\,. \end{aligned} \tag{5.6}$$

The solution is given by:

$$\begin{aligned} ds^2 &= \eta_{\mu\nu}dx^\mu dx^\nu + k\Lambda_k(x,\psi)\left[d\rho^2 + d\theta^2 + \frac{\tan^2\theta\, d\psi^2 + \tanh^2\rho\, d\varphi^2}{1+\tan^2\theta\,\tanh^2\rho}\right] \\ B &= \frac{k\Lambda_k(x,\psi)}{1+\tan^2\theta\,\tanh^2\rho}\, d\varphi\wedge d\psi + \frac{k\Lambda_k(x,\psi)\sin k\psi}{\tan\theta\,\sinh kx}\, d\varphi\wedge d\theta \\ e^{-2\phi} &= \frac{\alpha'}{k}U_0^2\Lambda_k^{-1}(x,\psi)\left(\cos^2\theta\cosh^2\rho + \sin^2\theta\sinh^2\rho\right). \end{aligned} \tag{5.7}$$

In the double scaling limit we have: $e^x = \cosh\rho/\sin\theta$.

5.2 The coset description

Our next task is to show that this supergravity background, eq. (5.7), corresponds actually to an exact coset conformal field theory. In the following we will consider the supersymmetric coset construction in order to compare directly with the previous results.

[11] We consider a special case of [13], where k stacks of N five-branes with a $\mathbb{Z}_k$ symmetry were considered. We restrict to the special case $N = 1$.

Superspace action for the coset. We consider the $SL(2,\mathbb{R})_k \times SU(2)_k$ superconformal theory. The supersymmetric WZW model is given by the following action for the matrix superfield $\mathcal{G}$, taking values in $SL(2,\mathbb{R}) \times SU(2)$:

$$kS(\mathcal{G}) = \frac{k}{2\pi}\int d^2z\, d^2\theta\, \mathrm{Tr}\left(\mathcal{G}^{-1}D\mathcal{G}\mathcal{G}^{-1}\bar{D}\mathcal{G} + \int dt\left\{\mathcal{G}\partial_t\mathcal{G}^{-1}\left(D\mathcal{G}\bar{D}\mathcal{G}^{-1} + \bar{D}\mathcal{G}D\mathcal{G}^{-1}\right)\right\}\right) \tag{5.8}$$

and the superfield $\mathcal{G}$ is parameterized in Euler angles for both group factors as follows:

$$\mathcal{G} = \mathrm{diag}\left(e^{\frac{i\sigma_2}{2}\Phi_L}e^{\frac{\sigma_1}{2}S}e^{\frac{i\sigma_2}{2}\Phi_R}, e^{\frac{i\sigma_2}{2}\tilde{\Phi}_L}e^{i\frac{\sigma_1}{2}\Omega}e^{\frac{i\sigma_2}{2}\tilde{\Phi}_R}\right), \tag{5.9}$$

where the superfields expanded in superspace-coordinates read:

$$\Phi_{L,R} = \phi_{L,R} + \theta\chi_{L,R} + \bar{\theta}\tilde{\chi}_{L,R} + \theta\bar{\theta}f_{L,R}, \tag{5.10}$$

and so on. We obtain the following action for $SL(2,\mathbb{R}) \times SU(2)$ at level k:

$$\begin{aligned} kS(\mathcal{G}) = \frac{k}{4\pi}\int d^2z\, d^2\theta\Big\{&DS\bar{D}S - D\Phi_L\bar{D}\Phi_L - D\Phi_R\bar{D}\Phi_R - 2\cosh S\; D\Phi_L\bar{D}\Phi_R \\ &+ D\Omega\bar{D}\Omega + D\tilde{\Phi}_L\bar{D}\tilde{\Phi}_L + D\tilde{\Phi}_R\bar{D}\tilde{\Phi}_R + 2\cos\Omega\, D\tilde{\Phi}_L\bar{D}\tilde{\Phi}_R\Big\}. \end{aligned} \tag{5.11}$$

We would like to gauge the following $U(1) \times U(1)$ null isometry of $SL(2,\mathbb{R}) \times SU(2)$:

$$(G, G') \to \left(e^{i\Lambda\sigma_2}Ge^{iM\sigma_2}, e^{i\Lambda\sigma_2}G'e^{iM\sigma_2}\right). \tag{5.12}$$

The isometry belongs to the elliptic subgroup of $SL(2,\mathbb{R})$.[12] Because the generator of the isometries in $SL(2,\mathbb{R}) \times SU(2)$ are null, the left and right gaugings are independent (see e.g. [48] for examples of null gaugings). Therefore the resulting target space will be four-dimensional. Note that in order to get a globally well-defined gauge theory, the gauge symmetry acts on $U(1)$ symmetries in $SL(2,\mathbb{R})$ and $SU(2)$ with the same periodicity. Therefore, one defines this coset with the *single cover* of $SL(2,\mathbb{R})$ rather than the universal cover which is usually considered.[13] The action of the gauged WZW model is given by adding the chiral projections of two $U(1)$ super-gauge fields, $\mathcal{A}$ and $\bar{\mathcal{A}}$:

$$\begin{aligned} kS(\mathcal{G}|\mathcal{A},\bar{\mathcal{A}}) = kS(\mathcal{G}) + \frac{k}{2\pi}\int d^2z\, d^2\theta\Big\{&\mathcal{A}\left(\bar{D}\Phi_R + \cosh S\,\bar{D}\Phi_L\right) + \bar{\mathcal{A}}\left(D\Phi_L + \cosh S\; D\Phi_R\right) - \\ &- \cosh S\,\mathcal{A}\bar{\mathcal{A}} + \mathcal{A}\left(\bar{D}\tilde{\Phi}_R + \cos\Omega\,\bar{D}\tilde{\Phi}_L\right) + \\ &+ \bar{\mathcal{A}}\left(D\tilde{\Phi}_L + \cos\Omega\, D\tilde{\Phi}_R\right) + \cos\Omega\,\mathcal{A}\bar{\mathcal{A}}\Big\}. \end{aligned} \tag{5.13}$$

Under the super-gauge transformations the superfields transform as follows:

$$\begin{aligned} \Phi_L &\to \Phi_L + \Lambda, & \Phi_R &\to \Phi_R + M \\ \tilde{\Phi}_L &\to \tilde{\Phi}_L + \Lambda, & \tilde{\Phi}_R &\to \tilde{\Phi}_R + M \\ \mathcal{A} &\to \mathcal{A} + D\Lambda, & \bar{\mathcal{A}} &\to \bar{\mathcal{A}} + \bar{D}M \end{aligned} \tag{5.14}$$

[12] Another gauging involving the hyperbolic subgroup gives the Nappi-Witten cosmological solution [47].

[13] Since the relative normalization of the action of the gauge fields in $SL(2,\mathbb{R})$ and $SU(2)$ is fixed by anomaly cancellation condition, the structure of the coset is rather rigid.

Therefore, we can make the super-gauge choice $\Phi_{L,R}=0$. We also define new coordinates: $\tilde{\Phi}_{L,R}=\Psi\pm\Phi$, $S=2R$ and $\Omega=2\Theta$. After integrating out the super-gauge fields, we find the following sigma-model:

$$S=\frac{k}{\pi}\int d^2z\,d^2\theta\Big\{DR\bar{D}R+D\Theta\bar{D}\Theta+\frac{\cos^2\Theta\cosh^2R\,D\Psi\bar{D}\Psi+\sin^2\Theta\sinh^2R\,D\Phi\bar{D}\Phi}{\cosh^2R-\cos^2\Theta}+ \\ +\frac{\sin^2\Theta\cosh^2R\left(D\Psi\bar{D}\Phi-\bar{D}\Psi D\Phi\right)}{\cosh^2R-\cos^2\Theta}\Big\}. \tag{5.15}$$

This sigma-model corresponds actually to the solution describing five-branes on a circle, eq. (5.7), in the limit $\Lambda_k\to 1$. A dilaton is generated by the change of the measure in the path integral, whose expression is given in the supergravity solution. Note that this sigma model receives no perturbative corrections, to all orders in $1/k$, because of $N=(4,4)$ superconformal symmetry. Non-perturbative corrections (i.e. worldsheet instantons) can however appear. Indeed, it has been remarked in [13] that the function Λ_k taking into account the discreteness of the distribution of five-branes, eq. (5.4), can actually be rewritten as the following infinite expansion:

$$\Lambda_k(x,\psi)=1+\sum_{m\neq 0}e^{-k(|m|x-im\psi)}\,, \tag{5.16}$$

which resembles an expansion in worldsheet instantons contributions. Therefore, as was done in [49] for the T-dual of ALF space (using complementary methods), it should be possible to capture the exact supergravity metric by considering instantons of the gauged WZW theory.

The link between this model and the $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)\times\mathrm{SU}(2)/\mathrm{U}(1)$ orbifold can be seen as follows: let us perform a duality transformation along the $\mathrm{U}(1)$ isometry generated by the superfield Ψ. Using the standard T-duality rules [50], one finds the following dual action:

$$S=\frac{k}{\pi}\int d^2z\,d^2\theta\Big\{DR\bar{D}R+D\Theta\bar{D}\Theta+\tanh^2R\frac{D\Psi\bar{D}\Psi}{k^2}+ \tag{5.17}$$
$$+\tan^2\Theta\left(D\Phi-\frac{D\Psi}{k}\right)\left(\bar{D}\Phi-\frac{\bar{D}\Psi}{k}\right)+R^{(2)}\ln(\cosh^2R\cos^2\Theta)\Big\}.$$

If we redefine $\Psi/k=\Psi'$ and shift the superfield Φ: $\Phi-\Psi/k=\Phi'$, we get exactly the superspace action for $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)\times\mathrm{SU}(2)/\mathrm{U}(1)$. We can have an indication of the presence of the $\mathbb{Z}_k$ orbifold for the following reason. The coordinate ψ has a periodicity: $\psi\sim\psi+2\pi n$, $n\in\mathbb{Z}$, inherited from the SU(2) group manifold. After the field redefinitions, the new coordinates have the periodicity: $\psi'\sim\psi'+2\pi n/k$, and $\varphi'\sim\varphi'+2\pi m-2\pi n/k$. Therefore we have an $\mathbb{Z}_k$ orbifold whose action is given by the identifications: $(\psi',\varphi')\sim(\psi'+2\pi n/k,\varphi'-2\pi n/k)$. This is the geometrical interpretation of the $\mathbb{Z}_k$ orbifold performed in section 4 to achieve spacetime supersymmetry. To further argue that the theories are indeed the same, we will use below the exact coset description we found to show that the spectrum of the coset $\{\mathrm{SL}(2,\mathbb{R})\times\mathrm{SU}(2)\}/(\mathrm{U}(1)_L\times\mathrm{U}(1)_R)$ and of the orbifold $\{\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)\times\mathrm{SU}(2)/\mathrm{U}(1)\}/\mathbb{Z}_k$ are explicitely the same.

Path integral analysis. We can use the standard techniques (see [30] and [51]) to obtain the spectrum of the CFT described by this supersymmetric coset. One starts with the path integral for the supersymmetric coset:

$$Z = \int [d\mathcal{G}][d\mathcal{A}][d\bar{\mathcal{A}}] e^{-kS(\mathcal{G}|\mathcal{A},\bar{\mathcal{A}})} . \tag{5.18}$$

Then one can parameterize the super-gauge fields by superfields taking values in the gauged U(1) subgroup H:

$$\mathcal{A} = \mathcal{H} D \mathcal{H}^{-1} , \qquad \bar{\mathcal{A}} = \tilde{\mathcal{H}} \bar{D} \tilde{\mathcal{H}}^{-1} , \tag{5.19}$$

and using the supersymmetric generalization of the Polyakov-Wiegmann identity [52], one can rewrite the path integral in a product form:

$$Z = \int [d\mathcal{G}][d\mathcal{H}][d\tilde{\mathcal{H}}][dB][dC] e^{-kS(\mathcal{H}^{-1}\mathcal{G}\tilde{\mathcal{H}})+kS(\mathcal{H}^{-1}\tilde{\mathcal{H}})-S_{SG}} , \tag{5.20}$$

with a super-ghosts action coming from the jacobian of the change of variables (5.19):

$$S_{SG} = \frac{1}{\pi} \int d^2z \, d^2\theta \, B\bar{D}C - \bar{B}D\bar{C} \tag{5.21}$$

with $B = \beta + \theta b$, $C = c + \theta\gamma$, and the same for the antiholomorphic superfields. Then one change the variable $\mathcal{H}^{-1}\mathcal{G}\tilde{\mathcal{H}} \to \mathcal{G}$ and gauge-fix: $\mathcal{H} = 1$. In the specific case of a null gauging we are considering, the super-WZW action for the null subgroup actually vanishes: $S(\tilde{\mathcal{H}}) = 0$. In short, the path integral for the model is just given by the sum of a $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SU}(2)$ piece and a superghost action:

$$Z = \int [d\mathcal{G}][dB][dC] e^{-kS(\mathcal{G})-S_{SG}} . \tag{5.22}$$

The theory possesses two chiral BRST currents. The BRST charge for the left-moving symmetry is, in terms of components:

$$\mathcal{Q}_{BRST} = \oint \frac{dz}{2\pi} \, : c\,(\mathcal{J}^3 + \mathcal{I}^3) : + : \gamma\,(\psi^3 + \chi^3) : \tag{5.23}$$

where $\mathcal{J}^3$ and $\mathcal{I}^3$ are respectively the total $\mathrm{SL}(2,\mathbb{R})$ and SU(2) currents at level k, and ψ^3, χ^3 are the associated fermions. We have a similar BRST charge for the right-movers.

5.3 The spectrum of the coset and comparison with the orbifold

The previous analysis shows that the spectrum of the coset is the spectrum of the $\mathrm{SL}(2,\mathbb{R}) \times$ SU(2) supersymmetric WZW model times the super-ghosts contribution, constrained by the BRST symmetry (5.23). For simplicity let us consider NS ground states. We recall that we have to consider the single cover of $\mathrm{SL}(2,\mathbb{R})$. The weights of the primaries of the coset will be then:

$$\begin{aligned} L_0^{cs} &= \frac{c_2}{k} + \frac{j'(j'+1)}{k} - \frac{w_L}{2} m + \frac{k}{4} w_L^2 \\ \bar{L}_0^{cs} &= \frac{c_2}{k} + \frac{j(j+1)}{k} - \frac{w_R}{2} \bar{m} + \frac{k}{4} w_R^2 \end{aligned} \tag{5.24}$$

where c_2 is the second Casimir of $SL(2,\mathbb{R})$ and $w_{L,R}$ the left and right spectral flows (see [25] for more details). In the BRST cohomology, the eigenvalues $m/2$ of $\mathcal{J}^3$, the $SL(2,\mathbb{R})$ Cartan, are constrained to be related to $m'/2$, those of $\mathcal{I}^3$, the $SU(2)$ Cartan:

$$\begin{aligned} m &= -m' \equiv j' - r' , \qquad r' = 0, \ldots, 2j \\ \bar{m} &= -\bar{m}' \equiv j' - \bar{r}' , \qquad \bar{r}' = 0, \ldots, 2j . \end{aligned} \tag{5.25}$$

Now let us refresh our memory about the spectrum of the primaries for the orbifold $\{SL(2,\mathbb{R})/U(1) \times SU(2)/U(1)\}/\mathbb{Z}_k$ considered in the previous sections. We also restrict our attention to states that are in the ground state of the NS sector. Then we have the following conformal dimensions:

$$\begin{aligned} L_0^{orb} &= \frac{c_2}{k} + \frac{j'(j'+1)}{k} + \frac{(m-\gamma-kw_+)^2}{4k} - \frac{m'^2}{4k} \\ \bar{L}_0^{orb} &= \frac{c_2}{k} + \frac{j(j+1)}{k} + \frac{(m+\gamma+kw_+)^2}{k} - \frac{(-m'+2\gamma)^2}{4k} , \end{aligned} \tag{5.26}$$

and due to the $\mathbb{Z}_k$ orbifold we have the constraint: $m + m' - \gamma = kp$. We do the change of variables: $-m' + 2\gamma \to \bar{m}'$, and find the spectrum:

$$\begin{aligned} L_0^{orb} &= \frac{c_2}{k} + \frac{j'(j'+1)}{k} + \frac{w_+ - p}{2}\, m' + \frac{k}{4}(w_+ - p)^2 \\ \bar{L}_0^{orb} &= \frac{c_2}{k} + \frac{j(j+1)}{k} + \frac{w_+ + p}{2}\, \bar{m}' + \frac{k}{4}(w_+ + p)^2 . \end{aligned} \tag{5.27}$$

To match the two spectra of primaries, eqs. (5.24) and (5.27), one has to identify the left and right spectral flows as follows:

$$w_L = w_+ - p , \qquad w_R = w_+ + p \tag{5.28}$$

and the matching of the complete spectra follows from the matching of the dimensions of the primaries, since the structure of the characters is the same.

To summarize, we have proven that the background corresponding to NS5-branes on a circle, eq. (5.7) is given by the worldsheet orbifold CFT $\{SL(2,\mathbb{R})/U(1) \times SU(2)/U(1)\}/\mathbb{Z}_k$. This was an assumption in [9], used to study little string theory in a double scaling limit. In this section we have given strong arguments for this equivalence, using an exact coset description of the background, and we explicitly matched the spectrum with the orbifold theory.

6. About spacetime supersymmetry and geometry

From this specific example we can draw important conclusions for the construction of supersymmetric string backgrounds in general. Suppose one would like to construct a superstring background that corresponds to the spacetime geometry of the two-dimensional euclidean black hole times some flat coordinates:

$$ds^2 = k(d\rho^2 + \tanh^2\rho\, d\varphi^2) + \eta_{\mu\nu} dx^\mu dx^\nu , \tag{6.1}$$

described by an $SL(2,\mathbb{R})_k/U(1)\times\mathbb{R}^{d,1}$ SCFT. Then one chooses an internal superconformal field theory such that the total central charge is $c = 15$. In our example, the "internal" CFT was simply $SU(2)_k/U(1)$, but one can choose more complicated examples as well. For the moment this theory cannot be spacetime supersymmeterized because the spectrum of charges of the $N = 2$ R-current is not integer. To achieve this goal, one has to perform a projection on integral charges, which is realized in string theory as an orbifold, because of the requirement of modular invariance. As in the example we studied in detail in this paper, the order of the orbifold scales like k. Therefore the effective geometry in which the low-energy effective theory lives is deeply altered. Let us illustrate this idea on a simple "toy model". We consider simply a $U(1)\times U(1)$ theory at level k. The spectrum of the zero modes is given by:[14]

$$\begin{aligned}\frac{L_0}{\alpha'} &= \frac{(n-kw)^2}{4\alpha' k}+\frac{(n'-kw')^2}{4\alpha' k}\\ \frac{\bar{L}_0}{\alpha'} &= \frac{(n+kw)^2}{4\alpha' k}+\frac{(n'+kw')^2}{4\alpha' k}\,.\end{aligned}$$

The spectrum of the low energy effective action is obtained in the limit $\alpha'\to 0$, with $\alpha' k$ fixed. In this limit, the left and right contributions are equal and corresponds to states with mass squared $m^2 = (n^2+n'^2)/(\alpha' k)$. The winding sectors are very massive and hence decouple from the low energy action. Now let us consider the $\mathbb{Z}_k$ diagonal orbifold of this theory. We have *(i)* a projection on the states with $n - n' \equiv kp$ (with p integer) and *(ii)* new twisted sectors. The resulting spectrum is:

$$\begin{aligned}\frac{L_0}{\alpha'} &= \frac{[n-k(w+\gamma/k)]^2}{4\alpha' k}+\frac{[n-k(w'+\gamma/k)+kp]^2}{4\alpha' k}\\ \frac{\bar{L}_0}{\alpha'} &= \frac{[n+k(w+\gamma/k)]^2}{4\alpha' k}+\frac{[n+k(w'+\gamma/k)+kp]^2}{4\alpha' k}\,.\end{aligned}$$

Then in the effective theory the contributions to the masses of the physical states will be:

$$\begin{aligned}m^2 &= 2\times\frac{(n-\gamma)^2}{4\alpha' k}\\ \bar{m}^2 &= 2\times\frac{(n+\gamma)^2}{4\alpha' k}\,.\end{aligned}$$

Note in particular that the twisted sectors survive to the low energy limit, precisely because the order of the orbifold is k. This is qualitatively the same behavior as in the $SL(2,\mathbb{R})/U(1)\times SU(2)/U(1)$ CFT considered in this paper.

Some years ago, the following paradox was pointed out (see [53, 54, 55]): some backgrounds of superstring theory should be supersymmetric, because of worldsheet extended superconformal symmetry. However, the corresponding supergravity solution is manifestly *not* supersymmetric. One of the most obvious examples which was considered was in fact the $SL(2,\mathbb{R})/U(1)\times SU(2)/U(1)$ background. The string frame metric and dilaton for this solution are:

$$ds^2 = k(d\rho^2+\tanh^2\rho\, d\varphi^2)+k(d\theta^2+\tan^2\theta\, d\psi^2)$$

[14] Only in this part, α' is reinstalled in the formulas instead of having being set to two.

$$e^{-2\phi} = \cosh^2\rho\cos^2\theta\,. \tag{6.2}$$

Let us consider for example the dilatino variation for type-IIB supergravity in this background. We have:

$$\delta\lambda = \left[\gamma^\mu\partial_\mu\phi\;\sigma_3 - \frac{1}{6}H_{\mu\nu\rho}\gamma^{\mu\nu\rho}\right]\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix} = \sqrt{k}\left[-\Gamma^\rho\tanh\rho + \Gamma^\theta\tan\theta\right]\sigma_3\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix}, \tag{6.3}$$

and therefore all the supersymmetry is explicitely broken.

We now argue for a resolution of this paradox. The requirement of spacetime supersymmetry enforces the projection on integer charges in the underlying worldsheet CFT, hence the $\mathbb{Z}_k$ orbifold. Therefore the supergravity solution (6.2) is *not* the correct low-energy effective background for this theory, since the spectrum we would get for supergravity fluctuations around the background (6.2) wouldn't match the $\alpha' \to 0$ limit (with $\alpha' k$ fixed) of the $\{\mathrm{SL}(2,\mathbb{R})/\,\mathrm{U}(1)\times\mathrm{SU}(2)/\,\mathrm{U}(1)\}\,/\mathbb{Z}_k$ spectrum. Rather the supergravity solution corresponding to this superstring background is given by the solution of five-branes on a circle, eq. (5.7), which is manifestly supersymmetric.

7. Conclusions

In this work we have computed the torus amplitude for the $\mathrm{SL}(2,\mathbb{R})_k/\,\mathrm{U}(1)$ supersymmetric coset. We have used a supersymmetric marginal deformation method to obtain this result from the supersymmetric $\mathrm{SL}(2,\mathbb{R})$ partition function, in analogy to [18] for bosonic SU(2) and [17] for bosonic $\mathrm{SL}(2,\mathbb{R})$. This is the first time this method is used for a supersymmetric WZW model. Then we have performed the exact decomposition in characters — and not only for the primary states — of the supersymmetric coset, which are the same as the irreducible characters of the $N = 2$ algebra with $c > 3$. This last fact is an important element in favor of the duality between the $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{U}(1)$ coset and the $N = 2$ super-Liouville theory. Moreover we have seen that the *extended* characters of $N = 2$, which have nice modular transformation properties, appear naturally in the decomposition. The spectrum we have found is composed of discrete representations, filling out exactly the improved range for the spin, and continuous representations accompanied by a divergent factor, due to the infinite volume available for them. Upon regularization, we find a divergent factor, corresponding to this volume divergence, and a finite part related to the reflection amplitude in $N = 2$ super-Liouville theory. Apart from these two pieces corresponding to standard characters of the coset, we find new sectors of states which may depend on the regularization procedure. Note that this is not a particular problem of our computation, but would also appear in the bosonic coset partition function [14] , the $\mathrm{SL}(2,\mathbb{R})$ partition function [17] (and presumably also in thermal AdS_3) if we also consider the *exact* expansion in characters, with the multiplicities, rather than only the expansion for the zero modes. This is an important open problem that deserves clarification.

Then we have considered the little string theory in a double scaling limit, corresponding to the worldsheet CFT $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{U}(1)\times\mathrm{SU}(2)/\,\mathrm{U}(1)$. The integrality of $N = 2$ charges — necessary to achieve spacetime supersymmetry — is obtained by considering a $\mathbb{Z}_k$ orbifold

JHEP06(2004)033

of this CFT, much as in Gepner models [3]. We provide the spacetime supersymmetric partition function for this superstring background. The computation of the Witten index agrees with the expected result for superstrings on an A_{k-1} ALE space. We also give some insight on the relationship between this theory and the theory of coincident NS5-branes, i.e. the $\mathrm{SU}(2)_k \times \mathbb{R}_Q$ worldsheet CFT. We show in particular that the leading part of the partition functions, proportional to the (infinite) volume agree.

We have shown that the supergravity solution for NS5-branes spread on a circle in the transverse space with $\mathbb{Z}_k$ symmetry, given in [13], is actually given by an exact CFT, corresponding to the null gauging: $(\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SU}(2))/(\mathrm{U}(1)_L \times \mathrm{U}(1)_R)$. After integrating out the gauge field, we obtain a sigma-model which is perturbatively exact, thanks to $N = (4,4)$ worldsheet superconformal symmetry. The metric, dilaton and NS-NS two form background fields agree with the supergravity solution, up to a multiplicative factor taking into account the discreteness of the distribution of branes. This factor could be interpreted as an expansion in worldsheet instantons contributions, that correct the sigma-model of the coset. Using the standard BRST quantization of gauged WZW model, we have shown that actually the spectrum of the null coset coincides precisely with the spectrum of the $(\mathrm{SL}(2,\mathbb{R})_k / \mathrm{U}(1) \times \mathrm{SU}(2)_k / \mathrm{U}(1))/\mathbb{Z}_k$ orbifold. This gives further strong arguments for the holographic duality between this worldsheet CFT and the double scaling limit of little string theory.

Getting inspiration from this example, we have discussed the relationship between spacetime supersymmetry of superstring backgrounds and low-energy effective geometry. To achieve spacetime supersymmetry with the $N = 2$ spectral flow one has to project onto integral charges of the $N = 2$ R-current, and this projection has a drastic effect on the interpretation of the corresponding supergravity background. We have illustrated this idea with a very simple $\mathrm{U}(1) \times \mathrm{U}(1)$ example and of course with the $\mathrm{SL}(2,\mathbb{R})_k / \mathrm{U}(1) \times \mathrm{SU}(2)_k / \mathrm{U}(1)$ model. The correct supergravity description of the corresponding supersymmetric string background is the solution for NS5-branes on a topologically trivial circle. Also in the non-supersymmetric cases, it is not clear when it is possible to make sense of a string theory with non-integer $N = 2$ charges since various OPE's between vertex operators of the theory will become non-local.

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JHEP06(2004)033